On chaotic Cartesian product of graphs and their retractions

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Abstract: Our aim in the present article is to introduce and study a new type of Cartesian product of graphs, namely chaotic Cartesian product of graphs.Chaotic graphs which represented as non trivial subgraphs of chaotic Cartesian product chaotic graphs are characterized. The chaotic incidence matrices and chaotic adjacency matrices representing the chaotic graphs induced from chaotic Cartesian product of graphs are obtained. The effect of retractions on a finite number of product chaotic subgarphs are deducede.

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1. Introduction

Chaos theory is applied in many scientific disciplines: mathematics, programming, microbiology, biology, computer, science, economics, engineering, finance, philosophy, physics, politics, population dynamics, psychology, and robotics. Chaotic behavior has been observed in the laboratory in a variety of systems including electrical circuits, lasers, oscillating chemical reactions, fluid dynamics, and mechanical and magneto-mechanical devices, as well as computer models of chaotic processes. Observations of chaotic behavior in nature include changes in weather, the dynamics of satellites in the solar system, the time evolution of the magnetic field of celestial bodies, population growth in ecology, the dynamics of the action potentials in neurons, and molecular vibrations. There is some controversy over the existence of chaotic dynamics in plate tectonics and in economics. One of the most successful applications of chaos theory has been in ecology, where dynamical systems such as the Ricker model have been used to show how population growth under density dependence can lead to chaotic dynamics. Chaos theory is also currently being applied to medical studies of epilepsy, specifically to the prediction of seemingly random seizures by observing initial conditions. A related field of physics called quantum chaos theory investigates the relationship between chaos and quantum mechanics. The correspondence principle states that classical mechanics is a special case of quantum mechanics, the classical limit. If quantum mechanics does not demonstrate an exponential sensitivity to initial conditions, it is unclear how exponential sensitivity to initial conditions can arise in practice in classical chaos. Recently, another field, called relativistic chaos, has emerged to describe systems that follow the laws of general relativity. The initial conditions of three or more bodies interacting through gravitational attraction can be arranged to produce chaotic motion[1-8,11,12].

An abstract graph G is a diagram consisting of a finite non -empty set of elements called vertices denoted by $V(G)$ together with a set of unordered pairs of these elements, called edges denoted by $E(G)$ [14,15]. A simple graph is a graph with no loops or multiple edges[14,15]. A connected graph is a graph in one piece [14,15]. Let G be a graph without loops with n -vertices labeled $1,2,3,\ldots,n$. The adjacency matrix $A(G)$ is the $n\times n$ matrix in which the entry in row i and column j is the number of edges joining the vertices i and j [14,15]. The incidence matrix of a graph gives the (0,1)-matrix which has a row for each vertex and column for each edge, and $(v,e)=1$ iff vertex v is incident e upon edge[14,15]. A graph H is said to be a subgraph of a graph G if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$ [9,13]. A subset A of a topological space X is called a retract of X if there exists a continuous map $r:X\to A$ (called a retraction) such that $r(a)=a,\forall a\in A$ [10].

2. Main results:

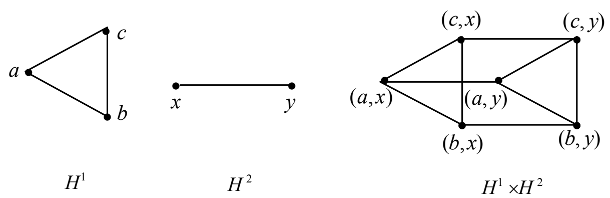


Figure 1. Cartesian product of two graphs

Aiming to our study we introduce the following:

Let H^1 and H^2 be two connected simple graphs of two vertex sets $\{a\ ,b,c\}\ ,\{x,y\}$. Then their Cartesian product is simple graph shown in Figure 1.

Both incidence and adjacency matrices I_1 , A_1 , I_2 , A_2 respectively of the graphs H^1 and H^2 take the forms

$$I_1=\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix},\ A_1=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

$$I_2=\begin{pmatrix}1&1\end{pmatrix},\ A_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

. On the other hand, both incidence and adjacency matrices of the Cartesian product $H^1\times H^2$ has of the following form

$$I=\begin{pmatrix}1&0&1&0&0&0\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&0&1&0&1\\0&0&0&1&1&0\\0&0&0&0&1&1\\0&0&1&0&0&1\\1&0&0&1&0&0\\0&1&0&0&1&0\end{pmatrix},\ A=\begin{pmatrix}0&1&1&1&0&0\\1&0&1&0&1&0\\1&1&0&0&0&1\\1&0&0&0&1&1\\0&1&0&1&0&1\\0&0&1&1&1&0\end{pmatrix}.$$

Then we arrive to the following theorem:

Theorem 1. The matrix of both incidence and adjacency represented the Cartesian product of two connected simple graphs can be obtained by the incidence and adjacency representing each of the given connected simple graphs constituting such a Cartesian product.

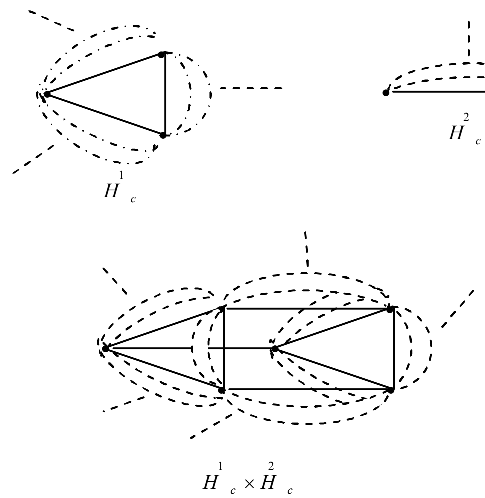


Figure 2. Chaotic Cartesian product.

Now we will discuss the Cartesian product of chaotic graphs. The chaotic graph is a geometric graph carry many other physical characters such that the chaotic graphs H_c^1 , H_c^2 , $H_c^1\times H_c^2$ shown in Figure 2.

Where

$H_c^1=(V(H_c^1),\ E(H_c^1))\ and\ H_c^2=(V(H_c^2),\ E(H_c^2))$,

$V_h^1=\{a_{ih},b_{ih},c_{ih},\ i=1,2,\ldots\}$,

$E_h^1=\{a_{ih}\,b_{ih}\ ,a_{ih}c_{ih}\ ,b_{ih}\,c_{ih}\ ,\ i=1,2,\ldots\}$,

$V_h^2=\{x_{ih},y_{ih},i=1,2,\ldots\}$ and

$E_h^2=\{x_{ih}\,y_{ih},i=1,2,\ldots\}$, where each pure chaotic vertex overlapped on the geometry. Both of the chaotic incidence and adjacency matrices of the of chaotic graphs H_c^1, H_c^2 are

$$I_c^1=\begin{pmatrix}\overset{c}{1}&\overset{c}{0}&\overset{c}{1}\\\overset{c}{1}&\overset{c}{1}&\overset{c}{0}\\\overset{c}{0}&\overset{c}{1}&\overset{c}{1}\end{pmatrix},\ A_c^1=\begin{pmatrix}\overset{c}{0}&\overset{c}{1}&\overset{c}{1}\\\overset{c}{1}&\overset{c}{0}&\overset{c}{1}\\\overset{c}{1}&\overset{c}{1}&\overset{c}{0}\end{pmatrix},\ I_c^2=\begin{pmatrix}\overset{c}{1}&\overset{c}{1}\end{pmatrix},\ A_c^2=\begin{pmatrix}\overset{c}{0}&\overset{c}{1}\\\overset{c}{1}&\overset{c}{0}\end{pmatrix}$$

where, $\overset{c}{1}=1_{0123\ldots\infty}$ and $\overset{c}{0}=0_{0123\ldots\infty}$. On the other hand, both chaotic incidence and adjacency matrices of the chaotic Cartesian product $H_c^1\times H_c^2$ has of the following forms

$$I_h=\begin{pmatrix} \overset{c}{1} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{0} \\ \overset{c}{1} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{0} & \overset{c}{0} \\ \overset{c}{0} & \overset{c}{1} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{0} \\ \overset{c}{0} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{1} \\ \overset{c}{0} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{1} & \overset{c}{0} \\ \overset{c}{0} & \overset{c}{0} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{1} \\ \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} \\ \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} \\ \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} \end{pmatrix},\; A_h=\begin{pmatrix} \overset{c}{0} & \overset{c}{1} & \overset{c}{1} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} \\ \overset{c}{1} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} \\ \overset{c}{1} & \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} \\ \overset{c}{1} & \overset{c}{0} & \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{1} \\ \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{1} & \overset{c}{0} & \overset{c}{1} \\ \overset{c}{0} & \overset{c}{0} & \overset{c}{1} & \overset{c}{1} & \overset{c}{1} & \overset{c}{0} \end{pmatrix}.$$

From the above discussion,we formulate the following theorem:

Theorem 2. The matrix of both incidence and adjacency represented the chaotic Cartesian product of two connected simple graphs can be obtained by the incidence and adjacency representing each of the given connected simple graphs constituting such chaotic Cartesian product.

Definition 1. The chaotic Cartesian product $H_c^1 \times H_c^2$ of chaotic connected simple graphs $H_c^1=(V(H_c^1),\, E(H_c^1))$ *and* $H_c^2=(V(H_c^2),\, E(H_c^2))$, is the chaotic graph with vertex set $(V(H_c^1)\,,V(H_c^2))$ where the chaotic vertex $(a_{ih}\,,x_{ih})$ is adjacent to vertex $(b_{ih}\,,y_{ih})$ $,i=1,2,\ldots,$ whenever $a_{ih}b_{ih}\in E(H_c^1)$ and $x_{ih}=y_{ih}$ or $a_{ih}=b_{ih}$ and $x_{ih}y_{ih}\in E(H_c^2)$. For fixed a chaotic vertex a_{ih} of H_c^1 the vertices $\{(a_{ih}\,,x_{ih}):x_{ih}\in V(H_c^2)\}$ induce chaotic subgraph of $H_c^1\times H_c^2$ is isomorphic to H_c^1.

Theorem 3. Let G_c be a connected simple graph x_{ih} its chaotic vertex.Then there exist a collection of chaotic subgraphs $H_c^1, H_c^2,\ldots,H_c^m$ of G_c, $x_{ih}\in V(H_c^j),\, j=1,2,\ldots m\,,i=1,2,\ldots$ such that $G_c=H_c^1\times H_c^2\times\cdots\times H_c^m$ induce a retraction $r(G_c)=H_c^1\times H_c^2\times\cdots\times r(H_c^s)\times\cdots\times H_c^m\,, for\; s=1,2,\ldots,m$ or $=H_c^1\times H_c^2\times\cdots\times r(H_c^s)\times\cdots\times r(H_c^t)\times\cdots\times H_c^m$, for $s,t=1,2,\ldots,m, s<t$ or $\cdots=\prod_{k=1}^{m} r(H_c^s)$.

proof: Consider the representation $G_c=H_c^1\times H_c^2\times\cdots\times H_c^m$ in which all chaotic subgraphs H_c^j are indecomoposable and $x_{ih}\in V(H_c^j)$, for all $j=1,2,\ldots m, i=1,2,\ldots$. We prove this result by mathematical induction on m, the number of chaotic subgraphs. When $m=2$, $r(H_c^1\times H_c^2)=r\{(V(H_c^1),E(H_c^1)\times(V(H_c^2),E(H_c^2)\}$ is the retraction of chaotic subgraphs with vertex set $(V(H_c^1)\,,V(H_c^2))$ where the chaotic vertex $(a_{ih}\,,x_{ih})$ is adjacent to vertex $(b_{ih}\,,y_{ih})\,,i=1,2,\ldots,$ whenever $a_{ih}b_{ih}\in E(H_c^1)$ and $x_{ih}=y_{ih}$ or $a_{ih}=b_{ih}$ and $x_{ih}y_{ih}\in E(H_c^2)$. Then clearly we get $r(H_c^1\times H_c^2)=r\{(V(H_c^1),E(H_c^1)\times(V(H_c^2),E(H_c^2)\}$
$=r\{(V(H_c^1),E(H_c^1)\}\times(V(H_c^2),E(H_c^2)$ or
$=(V(H_c^1),E(H_c^1))\times r\{(V(H_c^2),E(H_c^2)\}$ or
$=r\{(V(H_c^1),E(H_c^1)\}\times r\{(V(H_c^2),E(H_c^2)\}$.

Now suppose that the result is true for $(m-1)$ chaotic subgraphs, thus

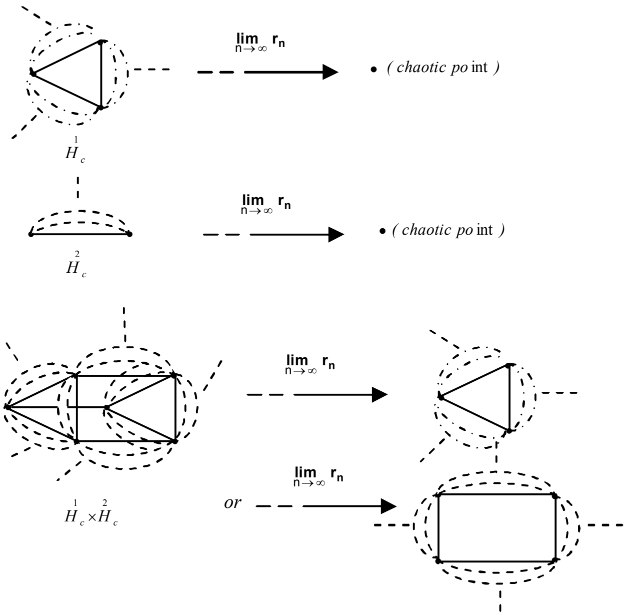
$r(H_c^1\times H_c^2\times\cdots\times H_c^{m-1}\times H_c^m)$
$=r(H_c^1\times H_c^2\times\cdots\times H_c^{m-1})\times r(H_c^m)$ or
$=r(H_c^1\times H_c^2\times\cdots\times H_c^{m-1})\times H_c^m$ or
$=H_c^1\times H_c^2\times\cdots\times H_c^{m-1}\times r(H_c^m)$. In case $r(H_c^1\times H_c^2\times\cdots\times H_c^{m-1}\times H_c^m)$
$=r(H_c^1\times H_c^2\times\cdots\times H_c^{m-1})\times r(H_c^m)$,we obtain from the induction assumption
$=r(H_c^1\times H_c^2\times\cdots\times H_c^{m-1})\times r(H_c^m)$
$=(H_c^1\times H_c^2\times\cdots\times r(H_c^s)\times\cdots\times r(H_c^{m-1}))\times r(H_c^m)$,
for $s=1,2,\ldots,m$ or
$=(H_c^1\times H_c^2\times\cdots\times r(H_c^s)\times\cdots\times r(H_c^t)\times\cdots\times r(H_c^{m-1}))$
$\times r(H_c^m), for\, s,t=1,2,\ldots,m, s<t$ or $\cdots$
or $=\prod_{k=1}^{m} r(H_c^s)$. Similarly for other two cases. This completes the proof.

Theorem 4. There are two chaotic connected simple subgraphs H_c^1 and H_c^2 such that their limit retraction of a chaotic Cartesian product is not isomorphic to the chaotic Cartesian product of limit retraction. i.e., $\lim_{n\to\infty} r_n(H_c^1\times H_c^2)\neq \lim_{n\to\infty} r_n(H_c^1)\times \lim_{n\to\infty} r_n(H_c^2)$.

proof: Let H_c^1 and H_c^2 be two chaotic connected simple subgraphs shown in Figure 3.

Figure 3. limit retractions.

Then $\lim_{n\to\infty} r_n(H_c^1 \times H_c^2)$ is either H_c^1 or homeomorphic to H_c^2. But $\lim_{n\to\infty} r_n(H_c^1) = \text{chaotic point}$. Therefore, $\lim_{n\to\infty} r_n(H_c^1 \times H_c^2) \neq \lim_{n\to\infty} r_n(H_c^1) \times \lim_{n\to\infty} r_n(H_c^2)$.



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