Riemann Hypothesis proved

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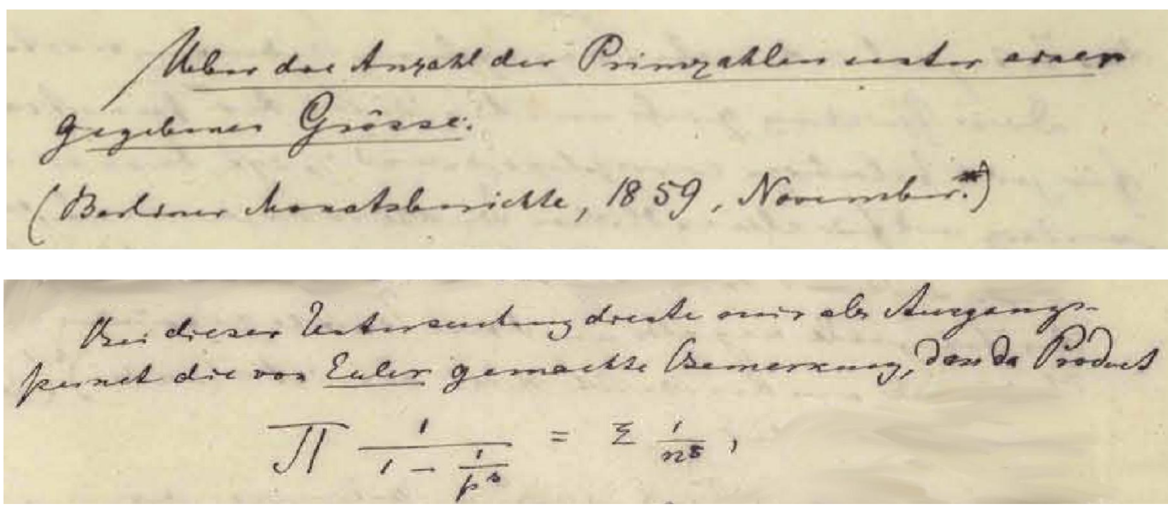
Abstract: We show a proof of the so-called Riemann Hypothesis (RH) stating that "All the non-trivial zero of the Zeta Function are on the Critical Line". We prove the RH using the theory of "inner product spaces" ***I*** and l^2 Hilbert spaces, where is defined the "functional" (a,b), named scalar [or inner] product of the vectors a and b. The proof is so simple that we suspect that there could be an error that we are unable to find.

[Galetto F. **Riemann Hypothesis proved.** *Academ Arena* 2014;6(12):19-22]. (ISSN 1553-992X). http://www.sciencepub.net/academia. 4

Key words: Zeta Function; zeros; Riemann Hypothesis

1. Introduction

It is well known, as said by many people, that for over a century mathematicians have been trying to prove the so-called Riemann Hypothesis, RH for short, a conjecture claimed by Riemann [who was professor at University of Gottingen in Germany], near 1859 in a 8-page paper "On the number of primes less than a given magnitude" shown at Berlin Academy, and dated/published in 1859; it is well known, as well, that RH is related to set of all the Prime Numbers.



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.*)

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

Figure 1 From the original B. Riemann manuscript

He invented the *Riemann zeta function*: ζ(z), where z is a complex number z=x+iy and i is the "positive" imaginary unit such that $i^2=-1$.

Before him Euler devised "the *Riemann zeta function*" for real numbers x>1, as "*Euler product*" (see figure 1); Riemann extended the function in the half-plane Re(z)=x>1 by the absolute convergent series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad (1)$$

and then, by analytic continuation, in the whole complex plane ***C***, with the exception of z=1 [harmonic series]: ζ(z) is a meromorphic function with only a simple pole at z=1 [with residue 1]; moreover ζ(z)≠0 for all z ∈ ***C*** with Re(z)=x>1; in particular ζ(1+ iy)≠0, for any y.

The function ζ(z) has zeros at the negative even integers **-2, -4, -6, …..**, named ***trivial zeros***.

The other zeros are named ***nontrivial*** *zeros*: all the <u>*"known"*</u> zeros, computed up to now [up to 2004, 10^{12} zeros have been computed, all on the *Critical Line*], are the complex numbers z=1/2 + iy, with suitable values of y.

Some properties of ζ(z) are as follows:

a) ζ(z) has no zero for Re(z)>1;

b) the only pole ζ(z) is at z=1=(1+i0): it is simple and has residue 1;

c) ζ(1+ iy)≠0, for any y≠0;

d) ζ(z) has the trivial zeros at the negative even integers z = -2, -4, -6, ….

e) all the nontrivial zeros lie inside the region, named *Critical Strip*, 0 ≤ Re(z) ≤ 1 and are symmetric about both the vertical line, named *Critical Line*, Re(z)=1/2 and the real axis Im(z)=0: $0=\zeta(z)=\zeta(\bar{z})=\zeta(1-z)=\zeta(1-\bar{z})$, as consequence of the functional equation (2) [which was proved by Riemann for all complex z (Riemann 1859).]

$$\pi^{-z/2}\Gamma(z/2)\zeta(z)=\pi^{-(1-z)/2}\Gamma[(1-z)/2]\zeta(1-z) \quad (2)$$

Riemann conjectured the so-called Riemann Hypothesis (RH): the <u>RH</u> states that <u>*All the nontrivial zeros of* ζ(z) *have real part x equal to 1/2 .*</u>

We will take advantage (in section 3 to prove RH) of the *"omega" function* ω(z)

$$\omega(z)=(1-2^{1-z})\zeta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z} \quad (3)$$

which has no pole, is convergent for Re(z) > 0 and have the same zeros of ζ(z) in the *Critical Strip*.

Many great mathematicians tackled this problem; we do not mention them, because they can be found in many books and papers.

<u>*If RH*</u> would be related to Physics, it would be considered a "universal law": up to 2004, 10^{13} zeros have been computed, all on the *Critical Line*.

<u>*If RH*</u> would be related to Statistics, <<the hypothesis H_0: "the nontrivial zeros are on the *Critical Line*">>, would be confirmed with a Confidence Level (CL) > 0.9999999999: the evidence of 10^{13} zeros computed, all on the *Critical Line* (as to 2004) supports H_0 with that "high" CL [if we could assume that the zeros are a "random sample"].

But Mathematics asks much more than Statistics and Physics…

Also a theorem of G. Hardy [*Hardy's Theorem, 1914*] who proved that "*There are infinitely many zeros of ζ(z) on the Critical Line*" is not enough because the theorem leave the possibility that other infinite zeros be in the *Critical Strip* and not on the Critical Line.

<u>*ALL*</u> the nontrivial zeros must be on the *Critical Line*, if one wants to prove RH.

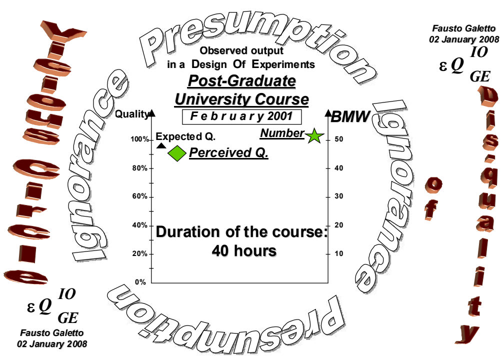


Figure 2 The Disquality Vicious Circle.

The author, Fausto Galetto, is aware that (in these weeks) he has been affording a very important problem that great mathematicians have failed to prove.

Having seen thousands of bad errors, he invented the *Vicious Circle of Disquality* [figure 2], to remind anybody (<u>himself included</u>) to use their own intelligence and rationality before making any statement…..

Since the proof of the RH, that he shows here, is very simple, he is worried very much that he was running in the Disquality Vicious Circle with his proof! Let's hope not…

2. Some concepts on "inner product spaces" and on Hilbert spaces

We have to ask the reader to refer to some Mathematics books for the ideas on "inner product spaces" and on Hilbert spaces. We remind here only that if W is an l^2 Hilbert space (on the field C of complex numbers) any vector

$$w=\{w_1, w_2, ..., w_n, \ldots\ldots\ldots\}$$

has the property that the series of the absolute squared components

$$\sum_{n=1}^{\infty}|w_n|^2 \quad (4)$$

is convergent.

In any "inner product space" ***I*** and in any Hilbert space ***H*** [also named l^2] is defined the "functional" <u>*(a,b)*</u>, named <u>*scalar*</u> [or <u>*inner*</u>] <u>*product of the vectors a and b*</u>, as the series [absolutely convergent]

$$(a,b)=\sum_{n=1}^{\infty}a_n\bar{b}_n \quad (5)$$

where a ∈ ***I*** [or ***H***] and b ∈ ***I*** [or l^2] are the points [vectors] and a_k and b_k [∈ ***C***] are the components of the vectors. According to the definition, the inner product (a,b) of two vectors a and b is a complex number.

When (a,b)=0 the vectors a and b are orthogonal.

We recall now that the *Riemann zeta function* is given the absolute convergent series, for x>1,

$$\zeta(x+iy)=\sum_{n=1}^{\infty}\frac{1}{n^{x+iy}} \quad (6)$$

and then extended analytically to the whole complex plane ***C***, where $\zeta(z)$ is a meromorphic function with only a simple pole at z=1 [with residue 1] (x= real part of z, y imaginary part of z).

From 3) and 6) we derive the following: when y=0 (and x≠0, in the *Critical Strip*) we have

$$\omega(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^x} \quad (7)$$

While, when x=0.5 (and y≠0, in the *Critical Strip*), we have

$$\omega(0.5+iy)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{0.5+iy}} \qquad (8)$$

If x > 0.5 ⇒ 2x > 1 then

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^{x}}\right|^{2}=\sum_{n=1}^{\infty}\left|\frac{1}{n^{2x}}\right|=\zeta(2x) \qquad (7b)$$

that is the norm ||α(x)||=$\sqrt{\zeta(2x)}$ where

α(x)={$1/1^x$, $-1/2^x$, $1/3^x$, $-1/4^x$,..., $(-1)^{n-1}/n^x$,……}=
={α_1, α_2, ..., α_n,………} (9)

Moreover, if x > 0.5 ⇒ 2x > 1 and if y≠0,

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^{x+iy}}\right|^{2}=\sum_{n=1}^{\infty}\left|\frac{1}{n^{2x+2iy}}\right|=\zeta(2x) \qquad (8b)$$

that is the norm ||β(x+iy)||=$\sqrt{\zeta(2x)}$ where

β(x+iy)={$1/1^{x+iy}$, $1/2^{x+iy}$, $1/3^{x+iy}$, $1/4^{x+iy}$,…., $1/n^{x+iy}$,…}={β_1, β_2, ..., β_n,……..} (10)

It is easily seen that α(x) ∈ ***l*** [or l^2], because 2x≥1 and the series for ζ(2x) is absolutely convergent, and β(x+iy) ∈ ***l*** [or l^2], because the series for ζ(2x+i2y) is absolutely convergent, as well. When 0 < x < 0.5 the norms ||α(x)|| and ||β(x+iy)|| are related to $\sqrt{|\zeta(2x)|}$; hence α(x) ∈ ***l*** and β(x+iy) ∈ ***l***, for 0< x<0.5, as well.

Now we consider d=x-0.5, and the infinite vectors
α(d)={$1/1^d$, $-1/2^d$, $1/3^d$, $-1/4^d$,...,$(-1)^{n-1}/n^d$,…}=
={α_1, α_2, ..., α_n,…} (9b)
β(iy)={$1/1^{iy+0.5}$, $1/2^{iy+0.5}$, $1/3^{iy+0.5}$, $1/4^{iy+0.5}$,…., $1/n^{iy+0.5}$,…}={β_1, β_2, ..., β_n,……..} (10)

So we can write the *"omega" function* as the scalar product

$$\omega(\bar{z})=(\alpha,\beta)=\sum_{n=1}^{\infty}\alpha_n\overline{\beta}_n \qquad (11)$$

The zeros of the *Riemann zeta function* ζ(z) are the same as the zeros of the *"omega" function* ω(z) that has no pole and is convergent for 0 < Re(z) < 1; the zeros of the *"omega" function* ω(z) are given by the vectors <u>*orthogonal*</u> in the ***l*** or l^2 spaces, such that

$$0=\omega(x-iy)=(\alpha,\beta)=\sum_{n=1}^{\infty}\alpha_n\overline{\beta}_n= \\ =\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{x-iy}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{0.5+d-iy}} \qquad (12)$$

Due to the functional equation and the fact that ζ(z) is analytic, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^{x+iy}}=\sum_{n=1}^{\infty}\frac{1}{n^{1-x+iy}}= \\ =\sum_{n=1}^{\infty}\frac{1}{n^{x-iy}}=\sum_{n=1}^{\infty}\frac{1}{n^{1-x-iy}} \qquad (13)$$

Taking advantage of the relationship between the *"omega" function* ω(z) and ζ(z) that have the same zeros in the *Critical Strip*, we can search for the zeros by finding the "vectors" α(d) and β(iy) satisfying

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{x+iy}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-x+iy}} \qquad (14)$$

that is the vectors orthogonal, for suitable values d and y,

α(d)={$1/1^d$, $-1/2^d$, $1/3^d$, $-1/4^d$,...,$(-1)^{n-1}/n^d$,…}
β(iy)={$1/1^{iy+0.5}$, $1/2^{iy+0.5}$, $1/3^{iy+0.5}$, …., $1/n^{iy+0.5}$,…}

3. The proof of Riemann Hypothesis

To prove that Riemann Hypothesis is true, <u>*we assume the hypothesis*</u> that RH is false, that is there are at least two points (zeros) z_1 and $1-\bar{z}_1$, **<u>not on</u>** the critical line, but symmetric to it, such that [taking advantage of the relationship between the *"omega" function* ω(z) and ζ(z)]

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{0.5+d_1+iy}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{0.5-d_1+iy}}=0 \qquad (15)$$

Actually there are 4 zeros, symmetric to the Critical Line and to the real axis, in the Critical Strip: z_1, $1-\bar{z}_1$, $z_2=-\bar{z}_1$, $1-\bar{z}_2$.

We search for the "vectors" satisfying (15); any couple of vectors
α(d)={$1/1^{d+0.5}$, $-1/2^{d+0.5}$, $1/3^{d+0.5}$,..., $(-1)^{n-1}/n^{d+0.5}$,…..}
α(-d)={$1/1^{0.5-d}$, $-1/2^{0.5-d}$, $1/3^{0.5-d}$,..., $(-1)^{n-1}/n^{0.5-d}$,…..}
intersect the origin (begins at 0 ∈ ***l*** [or l^2]); therefore either α(d) and α(-d) are coincident or they are different (and have an angle between them).

But α(d) and α(-d) are both orthogonal (from the origin 0 ∈ ***l*** [or l^2]) to the same vector
β(iy)={$1/1^{iy+0.5}$, $1/2^{iy+0.5}$, $1/3^{iy+0.5}$, …., $1/n^{iy+0.5}$,…}

This is impossible in spaces ***l*** [or l^2]) and therefore they must be the same vector: that is

α(d) = α(-d) ⇒ d=-d ⇒ d=0 ⇒ x=1-x ⇒ x=1/2

which contradicts our hypothesis that RH was false.

Since it was proved (Titchmarsh 1986) that there are infinite zeros of the *Riemann zeta function* ζ(z) in the Critical Strip, there are infinite values $z_k=x_k+iy_k$ such that $\zeta(z_k)=0=\omega(z_k)$, [0 < x_k < 1].

For any y_k such that $\zeta(z_k)=0$, either there is only one zero with $x_k=1/2$ (on the Critical Line) or two zeros, symmetric to the critical line, with different real parts x_k and $1-x_k$, that we proved, above, impossible for a particular value y_k such that $\zeta(z_k)=0=\omega(z_k)$.

Since we can repeat that for any y_k such that $\zeta(z_k)=0$, we have that $x_k=1/2$ for any nontrivial zero z_k: therefore ***RH is true***

It follows that, using the norm of such vectors $\alpha(x_k)$ and $\beta(iy_k)$, we have

$$\zeta(z_k)=0 \Rightarrow |\zeta(z_k)|=0 \Rightarrow \|\alpha(x_k)+\beta(iy_k)\|^2=\|\alpha(x_k)\|^2+\|\beta(iy_k)\|^2$$

that can be used also for computation purposes.

4. Conclusion

If the author did not run in the Disquality Vicious Circle with the previous proof of the Riemann Hypothesis, he has proved that RH is true…

On the contrary if he did run in the Disquality Vicious Circle with the previous proof, the Riemann Hypothesis is still unproved: we, please, ask the readers to inform Fausto Galetto of his errors.

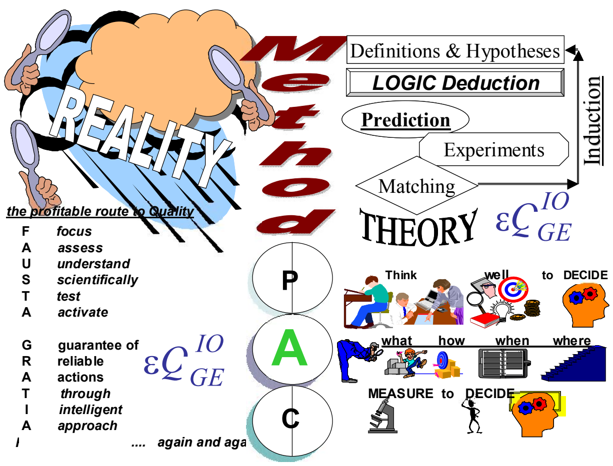


Figure 3 The Scientific Method

The Adult_ego_state (state A of "Transactional Analysis" of E. Berne, in the previous figure 3) is embodied in the εQ^{IO}_{GE} ***symbol*** used in the logo Fausto Galetto used while teaching Quality Management and Quality Methods at Turin Politecnico, [fig 4]

Figure 4 The epsilon_Quality of Intellectually hOnest people who use Gedanken Experimente

Intellectual hOnesty compels people to use as much as possible their rationality and Logic, in order not to deceive other people. Intellectually hOnest people speak to the Adult_ego_state of any other people (state A of "Transactional Analysis" of E. Berne, in the previous figure 3).

References

1. Titchmarsh E.C., The Theory of the Riemann Zeta-Function, CLARENDON PRESS, OXFORD, 1986

12/20/2014