Riemann Hypothesis solved

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Abstract: We show a proof of the so-called Riemann Hypothesis (RH) stating that "All the non-trivial zero of the Zeta Function are on the Critical Line". We prove the RH using the theory of l^2 Hilbert spaces. The proof is so simple that we suspect that there is an error that we are unable to find.

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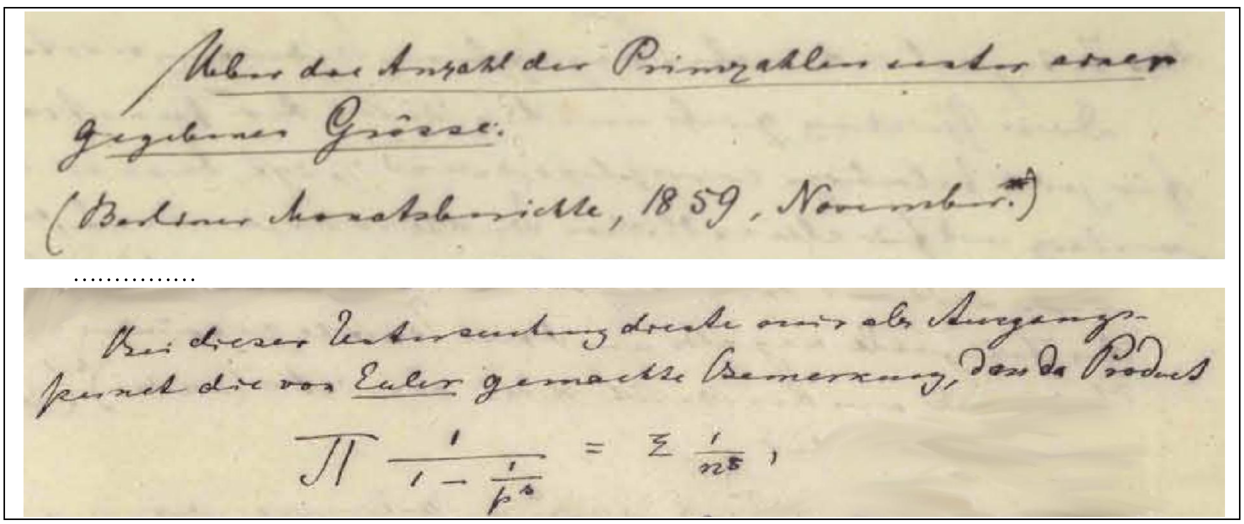
1. Introduction

It is well known that for over a century mathematicians have been trying to prove the so-called Riemann Hypothesis, RH for short, a conjecture claimed by Riemann [who was professor at University of Goettingen in Germany], near 1859 in a 8-page paper "On the number of primes less than a given magnitude" shown at Berlin Academy, and dated/published in 1859; it is well known, as well, that RH is related to set of all the Prime Numbers.



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.)

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Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

Fig 1 From the original B. Riemann manuscript.

He invented the *Riemann zeta function*: ζ(z), where z is a complex number z=x+iy and i is the "positive" imaginary unit such that $i^2=-1$.

The *Riemann zeta function*: is firstly defined in the half-plane Re(z)=x>1 by the absolute convergent series

1) $$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and then, by analytic continuation, in the whole complex plane ***C***, with the exception of z=1: $\zeta(z)$ is a meromorphic function with only a simple pole at z=1 [with residue 1]; moreover $\zeta(z)\neq 0$ for all z ∈ ***C*** with Re(z)=x>1.

It satisfies the functional equation

2) $$\pi^{-z/2}\,\Gamma(z/2)\zeta(z) = \pi^{-(1-z)/2}\,\Gamma[(1-z)/2])\zeta(1-z)$$

The function ζ(z) has zeros at the negative even integers **-2, -4, -6, …..**, named ***trivial zeros***.

The other zeros are named ***nontrivial*** *zeros*: all the *"known"* zeros, computed up to now [up to 2004, 10^{12}

zeros have been computed, all on the *Critical Line*], are the complex numbers z=1/2 + iy, with suitable values of y.

We can summarise the properties of ζ(z) as follows:

a) ζ(z) has no zero for Re(z)>1;

b) the only pole ζ(z) is at z=1: it is simple and has residue 1;

c) ζ(z) has trivial zeros at the negative even integers z = -2, -4, -6, ….

d) all the nontrivial zeros lie inside the region, named *Critical Strip*, 0 ≤ Re(z) ≤ 1 and are symmetric about both the vertical line, named *Critical Line*, Re(z)=1/2 and the real axis Im(z)=0: $\zeta(z)=\overline{\zeta(z)}=\zeta(\bar{z})=\zeta(1-z)=\zeta(1-\bar{z})$

Riemann conjectured the so-called Riemann Hypothesis (RH): the RH is

All nontrivial zeros of ζ(z) have real part x equal to 1/2 .

Since we can write

$$\omega(z)=(1-2^{1-z})\zeta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z} \tag{3}$$

we see that the *"omega" function* ω(z) has no pole and is convergent for 0 < Re(z) < 1; the importance of the *"omega" function* is due to the fact that ω(z) and ζ(z) have the same zeros in the *Critical Strip*.

We will use ω(z) in the next section to prove RH.

Many great mathematicians tackled this problem; we do not mention them, because they can be found in many books and papers.

If RH would be related to Physics, it would be considered a "universal law": up to 2004, 10^{12} zeros have been computed, all on the *Critical Line*.

If RH would be related to Statistics, <<the hypothesis H_0: "the nontrivial zeros are on the *Critical Line*">>, would be confirmed with a Confidence Level (CL) > 0.9999999999: the evidence of 10^{12} zeros computed, all on the *Critical Line* (as to 2004) supports H_0 with that "high" CL.

But Mathematics asks much more than Statistics and Physics…

Also a theorem of G. Hardy [*Hardy's Theorem, 1914*] that proved that "*There are infinitely many zeros of ζ(z) on the Critical Line*" is not enough.

ALL the nontrivial zeros must be on the *Critical Line*, if one wants to prove RH.

The author, Fausto Galetto, is aware that (in these weeks) he has been affording a very important problem that great mathematicians have failed to prove.

Having seen thousands of bad errors, he invented the *The Vicious Circle of Disquality*, to remind anybody (***himself included***) to use their own intelligence and rationality before making any statement…..

Since the proof of the RH, that he shows here, is very simple, he worries very much to be running in the Disquality Vicious Circle with his proof!!!!

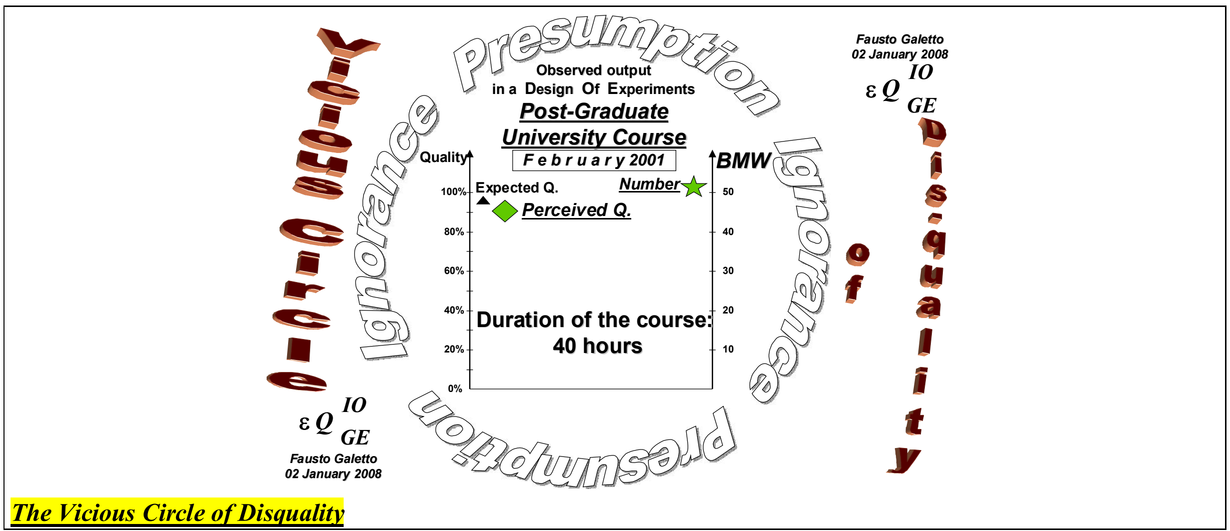

Fig 2 The Disquality Vicious Circle.

2. The proof of the Riemann Hypothesis

We have to ask the reader to refer to some Mathematics books for the ideas on Hilbert spaces. We remind here only that if W is an ℓ^2 Hilbert space (on the field C of complex numbers) any vector

w= {w_1, w_2, ..., w_n,……………..}

has the property that the series

$$\sum_{n=1}^{\infty} w_n^2 \tag{4}$$

is convergent.

Moreover, in any Hilbert space is defined the "functional" *(a,b)*, named *scalar product of the vectors a and b*, as the series [convergent]

5) $$(a,b) = \sum_{n=1}^{\infty} a_n \bar{b}_n$$

where a ∈ l^2 and b ∈ l^2 are the points and a_k and b_k [∈ ***C***] are the components of the vectors.

When (a,b)=0 the vectors a and b are orthogonal.

We recall now that the *Riemann zeta function* is given the absolute convergent series, for x>1,

6) $$\zeta(x+iy) = \sum_{n=1}^{\infty} \frac{1}{n^{x+iy}}$$

and then extended, by analytic continuation, to the whole complex plane ***C***, where $\zeta(z)$ is a meromorphic function with only a simple pole at z=1 [with residue 1] (x= real part of z, y imaginary part of z).

From 3) and 6) we derive the following: when y=0 (and x≠0, in the *Critical Strip*) we have

7) $$\omega(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x}$$

while when x=0 (and y≠0, in the *Critical Strip*) we have

8) $$\omega(iy) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{iy}}$$

We consider now the "infinite dimensions" vectors [points of the l^2 Hilbert space (on the field ***C***)]

9) $\alpha(x)=\{1/1^x, -1/2^x, 1/3^x, -1/4^x, \ldots\ldots\ldots, (-1)^{n-1}/n^x, \ldots\ldots\ldots\ldots\} = \{\alpha_1, \alpha_2, ..., \alpha_n, \ldots\ldots\ldots\ldots\}$

10) $\beta(iy)=\{1/1^{iy}, 1/2^{iy}, 1/3^{iy}, 1/4^{iy}, \ldots\ldots\ldots, 1/n^{iy}, \ldots\ldots\ldots\ldots\} = \{\beta_1, \beta_2, ..., \beta_n, \ldots\ldots\ldots\ldots\}$

It is easily seen that α(x) ∈ l^2, because ζ(2x) is not ∞, and β(y) ∈ l^2, because ζ(i2y) is not ∞, as well.

So we can write the *"omega" function* as the scalar product

11) $$\omega(\bar{z}) = (\alpha,\beta) = \sum_{n=1}^{\infty} \alpha_n \bar{\beta}_n$$

The zeros of the *Riemann zeta function* ζ(z) are the same as the zeros of the *"omega" function* ω(z) that has no pole and is convergent for 0 < Re(z) < 1; the zeros of the *"omega" function* ω(z) are given by the vectors *orthogonal* in the Hilbert space l^2, such that

12) $$0 = \omega(x-iy) = (\alpha,\beta) = \sum_{n=1}^{\infty} \alpha_n \bar{\beta}_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{x-iy}}$$

Due to the functional equation and the fact that ζ(z) is analytic, we have

13) $$\sum_{n=1}^{\infty} \frac{1}{n^{x+iy}} = \sum_{n=1}^{\infty} \frac{1}{n^{1-x-iy}} = \sum_{n=1}^{\infty} \frac{1}{n^{x-iy}} = \sum_{n=1}^{\infty} \frac{1}{n^{1-x+iy}}$$

Taking advantage of the relationship between the *"omega" function* ω(z) and ζ(z) that have the same zeros in the *Critical Strip*, we can search for the zeros by finding the "vectors" α(x) and β(iy) satisfying

14) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{x+iy}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-x+iy}} = 0$$

that is the vectors orthogonal, for SUITABLE values x and y,

$\alpha(x)=\{1/1^x,\ -1/2^x,\ 1/3^x,\ -1/4^x,\ldots\ldots.,\ (-1)^{n-1}/n^x,\ldots\ldots\ldots\}$

$\beta(iy)=\{1/1^{iy},\ 1/2^{iy},\ 1/3^{iy},\ 1/4^{iy},\ldots\ldots.,\ 1/n^{iy},\ldots\ldots\ldots\}$

To prove that RH is true, *we assume* that it is false.

Therefore there are at least two points (zeros) z_1 and $1-\bar{z}_1$, **not on** the critical line, but symmetric to it, such that

15) $$\sum_{n=1}^{\infty}\frac{1}{n^{x_1+iy}}=\sum_{n=1}^{\infty}\frac{1}{n^{1-x_1+iy}}=0$$

Actually there are 4 zeros, symmetric to the Critical Line and to the real axis, in the Critical Strip: $z_1,\ 1-\bar{z}_1,\ z_2=-\bar{z}_1,\ 1-\bar{z}_2$

We take advantage of the relationship between the *“omega” function* ω(z) and ζ(z); so we search for the “vectors” satisfying

16) $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{x_1+iy}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-x_1+iy}}=0$$

Any couple of vectors

$\alpha(x)=\{1/1^x,\ -1/2^x,\ 1/3^x,\ -1/4^x,\ldots\ldots.,\ (-1)^{n-1}/n^x,\ldots\ldots\ldots\}$

$\alpha(1-x)=\{1/1^{1-x},\ -1/2^{1-x},\ 1/3^{1-x},\ -1/4^{1-x},\ldots\ldots.,\ (-1)^{n-1}/n^{1-x},\ldots\ldots\ldots\}$

intersect the origin (begins at $0 \in \boldsymbol{l}^2$) of the Hilbert space $\boldsymbol{l}^2$; therefore either α(x) and α(1-x) are coincident or they are different (and have an angle between them).

But α(x) and α(1-x) are both orthogonal (from the origin of the Hilbert space $\boldsymbol{l}^2$) to the same vector

$\beta(iy)=\{1/1^{iy},\ 1/2^{iy},\ 1/3^{iy},\ 1/4^{iy},\ldots\ldots.,\ 1/n^{iy},\ldots\ldots\ldots\}$

This is impossible in the Hilbert space $\boldsymbol{l}^2$ and therefore they must be the same vector: that is

$\alpha(x) = \alpha(1-x) \Rightarrow x=1-x \Rightarrow x=1/2$

which contradicts our hypothesis that RH was false.

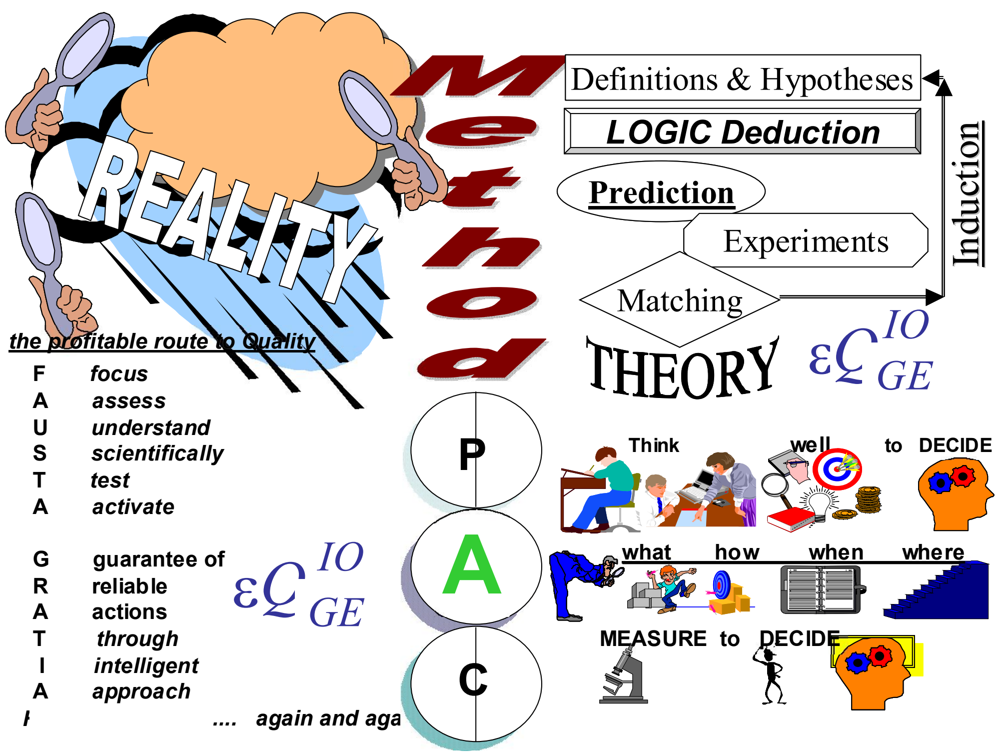

Fig 3 The Scientific Method

Since it was proved that there are infinite zeros of the *Riemann zeta function* ζ(z) in the Critical Strip, there are infinite values $z_k=x_k+iy_k$ such that $\zeta(z_k)=0=\omega(z_k)$, $[0 < x_k < 1]$.

For any y_k such that $\zeta(z_k)=0$, either there is only one zero with $x_k=1/2$ (on the Critical Line) or two zeros, symmetric to the critical line, with different real parts x_k and $1-x_k$, that we proved, above, impossible for a particular value y_k such that $\zeta(z_k)=0=\omega(z_k)$.

Since we can repeat that for any y_k such that $\zeta(z_k)=0$, we have that $x_k=1/2$ for any nontrivial zero z_k: ***RH is true***

It follows that, using the norm of such vectors $\alpha(x_k)$ and $\beta(iy_k)$, we have

$$\zeta(z_k)=0 \Rightarrow |\zeta(z_k)|=0 \Rightarrow \|\alpha(x_k)+\beta(iy_k)\|^2= \|\alpha(x_k)\|^2 + \|\beta(iy_k)\|^2$$

that can be used also for computation purposes.

3. Conclusion

If the author did not run in the Disquality Vicious Circle with the previous proof of the Riemann Hypothesis, he has proved that RH is true…

On the contrary if we did run in the Disquality Vicious Circle with the previous proof, the Riemann Hypothesis is still unproved: we ask the readers to inform Fausto Galetto of his errors.

The Adult_ego_state (state A of "Transactional Analysis" of E. Berne, in the previous figure 3) is embodied in the εQ^{IO}_{GE} ***symbol*** [fig 4]

12/1/2014

Fig 4. The epsilon_Quality of Intellectually hOnest people who use Gedanken Experimente

Intellectual hOnesty compels people to use as much as possible their rationality and Logic, in order not to deceive other people. Intellectually hOnest people speak to the Adult_ego_state of other people (state A of "Transactional Analysis" of E. Berne, in the previous figure 3).

References

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