The Hardy-Littlewood prime *k*-tuple conjecture is false

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Abstract: Using Jiang function we prove Jiang prime k-tuple theorem. We prove that the Hardy-Littlewood prime k-tuple conjecture is false. Jiang prime k-tuple theorem can replace the Hardy-Littlewood prime k-tuple conjecture.

Keywords: Jiang function; prime k-tuple theorem; Hardy-Littlewood prime; k-tuple conjecture

(A) Jiang prime k-tuple theorem [1, 2].

We define the prime k-tuple equation

$$p, p + n_i \tag{1}$$

where $2|n_i, i = 1, \cdots k-1$.

we have Jiang function [1, 2]

$$J_2(\omega) = \prod_P (P - 1 - \chi(P)) \tag{2}$$

where $\omega = \prod_P P$, $\chi(P)$ is the number of solutions of congruence

$$\prod_{i=1}^{k-1}(q + n_i) \equiv 0 \pmod{P}, \quad q = 1, \cdots, p-1 \tag{3}$$

If $\chi(P) < P-1$ then $J_2(\omega) \neq 0$. There exist infinitely many primes P such that each of $P + n_i$ is prime. If $\chi(P) = P-1$ then $J_2(\omega) = 0$. There exist finitely many primes P such that each of $P + n_i$ is prime. $J_2(\omega)$ is a subset of Euler function $\phi(\omega)$ [2].

If $J_2(\omega) \neq 0$, then we have the best asymptotic formula of the number of prime P [1, 2]

$$\pi_k(N, 2) = \left|\{P \leq N : P + n_i = prime\}\right| \sim \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N} = C(k)\frac{N}{\log^k N} \tag{4}$$

$$\phi(\omega) = \prod_P (P-1),$$

$$C(k) = \prod_P \left(1 - \frac{1+\chi(P)}{P}\right)\left(1 - \frac{1}{P}\right)^{-k} \tag{5}$$

Example 1. Let $k = 2, P, P + 2$, twin primes theorem.
From (3) we have

$$\chi(2) = 0, \quad \chi(P) = 1 \text{ if } P > 2, \tag{6}$$

Substituting (6) into (2) we have

$$J_2(\omega) = \prod_{P \geq 3}(P-2) \neq 0$$

(7)

There exist infinitely many primes P such that $P+2$ is prime. Substituting (7) into (4) we have the best asymptotic formula

$$\pi_k(N,2) = \left|\{P \leq N : P+2 = prime\}\right| \sim 2 \prod_{P \geq 3}(1 - \frac{1}{(P-1)^2})\frac{N}{\log^2 N}.$$

(8)

Example 2. Let $k=3, P, P+2, P+4$.

From (3) we have

$$\chi(2) = 0, \quad \chi(3) = 2$$

(9)

From (2) we have

$$J_2(\omega) = 0.$$

(10)

It has only a solution $P=3$, $P+2=5$, $P+4=7$. One of $P, P+2, P+4$ is always divisible by 3.

Example 3. Let $k=4, P, P+n$, where $n=2,6,8$.

From (3) we have

$$\chi(2) = 0, \chi(3) = 1, \chi(P) = 3 \text{ if } P > 3.$$

(11)

Substituting (11) into (2) we have

$$J_2(\omega) = \prod_{P \geq 5}(P-4) \neq 0,$$

(12)

There exist infinitely many primes P such that each of $P+n$ is prime.

Substituting (12) into (4) we have the best asymptotic formula

$$\pi_4(N,2) = \left|\{P \leq N : P+n = prime\}\right| \sim \frac{27}{3} \prod_{P \geq 5} \frac{P^3(P-4)}{(P-1)^4} \frac{N}{\log^4 N}$$

(13)

Example 4. Let $k=5$, P, $P+n$, where $n=2,6,8,12$.

From (3) we have

$$\chi(2) = 0, \chi(3) = 1, \chi(5) = 3, \chi(P) = 4 \text{ if } P > 5$$

(14)

Substituting (14) into (2) we have

$$J_2(\omega) = \prod_{P \geq 7}(P-5) \neq 0$$

(15)

There exist infinitely many primes P such that each of $P+n$ is prime. Substituting (15) into (4) we have the best asymptotic formula

$$\pi_5(N,2) = \left|\{P \leq N : P+n = prime\}\right| \sim \frac{15^4}{2^{11}} \prod_{P \geq 7} \frac{(P-5)P^4}{(P-1)^5} \frac{N}{\log^5 N}$$

(16)

Example 5. Let $k=6$, P, $P+n$, where $n=2,6,8,12,14$.

From (3) and (2) we have

$$\chi(2) = 0, \chi(3) = 1, \chi(5) = 4, \quad J_2(5) = 0$$

(17)

It has only a solution $P=5$, $P+2=7$, $P+6=11$, $P+8=13$, $P+12=17$, $P+14=19$. One of $P+n$ is always divisible by 5.

（B）**The Hardy-Littlewood prime k-tuple conjecture[3-14].**
This conjecture is generally believed to be true, but has not been proved(Odlyzko et al.1999).

We define the prime k-tuple equation

$$P, P + n_i \tag{18}$$

where $2|n_i, i = 1,\cdots, k-1$.

In 1923 Hardy and Littlewood conjectured the asymptotic formula

$$\pi_k(N,2) = \left|\{P \le N : P + n_i = prime\}\right| \sim H(k)\frac{N}{\log^k N}, \tag{19}$$

where

$$H(k) = \prod_P \left(1 - \frac{v(P)}{P}\right)\left(1 - \frac{1}{P}\right)^{-k} \tag{20}$$

$v(P)$ is the number of solutions of congruence

$$\prod_{i=1}^{k-1}(q + n_i) \equiv 0 \pmod{P}, \qquad q = 1,\cdots, P. \tag{21}$$

From (21) we have $v(P) < P$ and $H(k) \ne 0$. For any prime k-tuple equation there exist infinitely many primes P such that each of $P + n_i$ is prime, which is false.

Conjecture 1. Let $k = 2, P, P + 2$, twin primes theorem

From (21) we have

$$v(P) = 1 \tag{22}$$

Substituting (22) into (20) we have

$$H(2) = \prod_P \frac{P}{P-1} \tag{23}$$

Substituting (23) into (19) we have the asymptotic formula

$$\pi_2(N,2) = \left|\{P \le N : P + 2 = prime\}\right| \sim \prod_P \frac{P}{P-1}\frac{N}{\log^2 N} \tag{24}$$

which is false see example 1.

Conjecture 2. Let $k = 3, P, P + 2, P + 4$.

From (21) we have

$$v(2) = 1, \ v(P) = 2 \text{ if } P > 2 \tag{25}$$

Substituting (25) into (20) we have

$$H(3) = 4 \prod_{P \ge 3} \frac{P^2(P-2)}{(P-1)^3} \tag{26}$$

Substituting (26) into (19) we have asymptotic formula

$$\pi_3(N,2) = \left|\{P \le N : P + 2 = prime, P + 4 = prim\}\right| \sim 4\prod_{P \ge 3}\frac{P^2(P-2)}{(P-1)^3}\frac{N}{\log^3 N} \tag{27}$$

which is false see example 2.

Conjecture 3. Let $k = 4$, $P, P + n$, where $n = 2, 6, 8$.

From (21) we have

$$\nu(2) = 1, \ \nu(3) = 2, \ \nu(P) = 3 \quad \text{if } P > 3 \tag{28}$$

Substituting (28) into (20) we have

$$H(4) = \frac{27}{2} \prod_{P>3} \frac{P^3(P-3)}{(P-1)^4} \tag{29}$$

Substituting (29) into (19) we have asymptotic formula

$$\pi_4(N, 2) = \left|\{P \le N : P + n = prime\}\right| \sim \frac{27}{2} \prod_{P>3} \frac{P^3(P-3)}{(P-1)^4} \frac{N}{\log^4 N} \tag{30}$$

Which is false see example 3.

Conjecture 4. Let $k = 5$, $P, P + n$, where $n = 2, 6, 8, 12$

From (21) we have

$$\nu(2) = 1, \ \nu(3) = 2, \ \nu(5) = 3, \ \nu(P) = 4 \quad \text{if } P > 5 \tag{31}$$

Substituting (31) into (20) we have

$$H(5) = \frac{15^4}{4^5} \prod_{P>5} \frac{P^4(P-4)}{(P-1)^5} \tag{32}$$

Substituting (32) into (19) we have asymptotic formula

$$\pi_5(N, 2) = \left|\{P \le N : P + n = prime\}\right| \sim \frac{15^4}{4^5} \prod_{P>5} \frac{P^4(P-4)}{(P-1)^5} \frac{N}{\log^5 N} \tag{33}$$

Which is false see example 4.

Conjecture 5. Let $k = 6$, P, $P + n$, where $n = 2, 6, 8, 12, 14$.

From (21) we have

$$\nu(2) = 1, \ \nu(3) = 2, \ \nu(5) = 4, \ \nu(P) = 5 \quad \text{if } P > 5 \tag{34}$$

Substituting (34) into (20) we have

$$H(6) = \frac{15^5}{2^{13}} \prod_{P>5} \frac{(P-5)P^5}{(P-1)^6} \tag{35}$$

Substituting (35) into (19) we have asymptotic formula

$$\pi_6(N, 2) = \left|\{P \le N : P + n = prime\}\right| \sim \frac{15^5}{2^{13}} \prod_{P>5} \frac{(P-5)P^5}{(P-1)^6} \frac{N}{\log^6 N} \tag{36}$$

which is false see example 5.

Conclusion.

The Hardy-Littlewood prime k-tuple conjecture is false. The tool of additive prime number theory is basically the Hardy-Littlewood prime k-tuple conjecture. Jiang prime k-tuple theorem can replace Hardy-Littlewood prime k-tuple Conjecture. There cannot be really modern prime theory without Jiang function.

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