Derivations of Tensor Product of Finite Number of Simple C*-Algebras.

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Abstract: In this paper we construct the derivations of $\overset{n}{\underset{i=1}{\otimes}} A_i$ in terms of the derivations of some simple C^*-algebras $A_i \ \forall i = 1,2,.......,n.$ Also we introduce the concept of relative compatibility of finite number of A_i - derivations $\forall i = 1,2,......,n.$ We express the general form of any element c in the kernel of $\delta_{x \otimes \otimes_{k=2}^{n} I}$ where $c \in \otimes_{i=1}^{n} A_i$ and $x \in A_1$ in terms of some simple tensor product $c = I \otimes \otimes_{k=2}^{n} b_k$, $b \in \otimes_{k=2}^{n} A_k$. Finally we get a precise form of $A_i -$ derivations $(\delta_i) \quad \forall i = 1,2,3,.....,n$ in terms of a sequence of derivation $(\xi_{i_j})_{j=1}^{\infty}$ on A_i and their basis $(e_{i_j})_{j=1}^{\infty} \quad \forall i = 1,2,3,.....,n$ .For resent results see [1],[3],[5] and [10]. [Journal of American Science 2010;6(8):31-38]. (ISSN: 1545-1003).

Key words: Simple C^*-algebra; Tensor product of C^*-algebra; A-derivation; Compatible derivation.

1. Introduction

If $d : \overset{n}{\underset{i=1}{\otimes}} A_i \to \overset{n}{\underset{i=1}{\otimes}} A_i$ is a derivation where $\overset{n}{\underset{i=1}{\otimes}} A_i$ be a tensor product of finite number of simple C^*- algebras.

Then for $\left(\overset{n}{\underset{i=1}{\otimes}} a_i\right), \left(\overset{n}{\underset{i=1}{\otimes}} b_i\right) in \left(\overset{n}{\underset{i=1}{\otimes}} A_i\right)$, we have

$$d\left(\left(\overset{n}{\underset{i=1}{\otimes}} a_i\right)\left(\overset{n}{\underset{i=1}{\otimes}} b_i\right)\right) = d\left(\overset{n}{\underset{i=1}{\otimes}} a_i\right)\left(\overset{n}{\underset{i=1}{\otimes}} b_i\right) + \left(\overset{n}{\underset{i=1}{\otimes}} a_i\right) d\left(\overset{n}{\underset{i=1}{\otimes}} b_i\right).$$

Let I be the identity of $A_i \ \forall i = 1,....,n.$ By $A_i \quad (\forall i = 1,2,......,n)$ we shall always mean simple C^*- algebras with countable basis. Each simple C^*- algebra A_i has the property $¢(A_i) = CI$, where $¢(A_i)$ is the centre of A_i see [9]. Recall that by a simple C^*- algebra A_i ,we shall always mean a C^*- algebras whose ideals are $\{0\}$ and A.

A linear map $D_i : A_i \to A_i, \quad \forall i = 1.2,.....,n$ is called a derivation if for each $a_i, b_i \in A_i$

$$D_i(a_i b_i) = D_i(a_i) b_i + a_i D_i(b_i).$$

It is called a*-derivation if it satisfies $D_i(a_i)^* = D_i(a_i^*) \ \forall 1 \le i \le n$.

For a fixed element $a_i \in A_i$, we can define $D_{a_i} : A_i \to A_i$,

where $D_{a_i}(b_i) = [a_i, b_i] = a_i b_i - b_i a_i$. It is known that D_{a_i} is a derivation,

which is called an inner derivation [9].

A derivation D_{a_i} is called approximately inner if it is the limit of a sequence of inner derivations. For more details about the definitions and results we can refer to [7]and [8].

2. Compatible derivations of finite number of simple C*- algebras.

Now we are going to define a derivation of finite number of simple C*- algebra.

Definition 2.1

Let $\overset{n}{\underset{i=1}{\otimes}} A_i$ be a tensor product of finite number of simple C^*- algebras. A linear map $\delta_i : A_i \to \overset{n}{\underset{i=1}{\otimes}} A_i$ is called an A_i - derivation with respect to $\overset{n}{\underset{i=1}{\otimes}} A_i$

, $i = 1, 2, \ldots, n$ if it satisfies,

$$\delta_1(a_1 b_1) = \delta_1(a_1)\left(b_1 \otimes (\otimes_{k=2}^{n} I)\right) + \left(a_1 \otimes (\otimes_{k=2}^{n} I)\right)\delta_1(b_1),$$

$$\delta_i(a_i b_i) = \delta_i(a_i)\left(\otimes(\otimes_{k=1}^{i-1} I) \otimes b_i \otimes (\otimes_{k=i+1}^{n} I)\right) + \left(\otimes(\otimes_{k=1}^{i-1} I) \otimes a_i \otimes \otimes_{k=i+1}^{n} I\right)\delta_i(b_i), \quad 2 \le i \le n-1.$$

$$\delta_n(a_n b_n) = \delta_n(a_n)\left((\otimes_{k=1}^{n-1} I) \otimes b_n\right) + \left((\otimes_{k=1}^{n-1} I) \otimes a_n\right)\delta_n(b_n).$$

It is called a * - derivation with respect to $\overset{n}{\underset{i=1}{\otimes}} A_i$, if

$$\delta_i(a_i)^* = \delta_i(a_i^*) \quad \forall_i = 1, 2, .., n.$$

Example 2.2

Let $C \in \overset{n}{\underset{i=1}{\otimes}} A_i$ and define

$\delta_i : A_i \to \overset{n}{\underset{i=1}{\otimes}} A_i \ \ i = 1, 2, \ldots, n$ by

$$\delta_1(a_1) = \delta_c\left(a_1 \otimes \overset{n}{\underset{k=2}{\otimes}} I\right)$$

$$\delta_i(a_i) = \delta_c\left(\left(\overset{i-1}{\underset{k=1}{\otimes}} I\right) \otimes a_i \otimes \left(\overset{n}{\underset{k=i+1}{\otimes}} I\right)\right)$$

$\forall i = 2, \ldots, n-1,$

and $\delta_n(a_n) = \delta_c\left(\overset{n-1}{\underset{k=1}{\otimes}} I \otimes a_n\right)$.Then δ_i is an A_i - derivation with respect to $\overset{n}{\underset{i=1}{\otimes}} A_i \ \ \forall i = 1, 2, \ldots, n$,

which is called an inner A_i - derivation.

Next we are introduce the notion of compatibility of A_i - derivations $\forall i = 1, 2, \ldots, n$.

Definition 2.3

Let δ_i be A_i - derivation with respect to $\overset{n}{\underset{i=1}{\otimes}} A_i$, $i = 1, 2, \ldots, n$, then δ_i 's are compatible if the map

$d_i : \overset{n}{\underset{i=1}{\otimes}} A_i \to \overset{n}{\underset{i=1}{\otimes}} A_i$ defined by

$$d\left(\overset{n}{\underset{i=1}{\otimes}} a_i\right) = \delta_1(a_1)\left(I \otimes \overset{n}{\underset{k=2}{\otimes}} a_k\right) + \sum_{i=2}^{n-1}\left(\overset{i-1}{\underset{k=1}{\otimes}} a_k \otimes \overset{n}{\underset{k=i}{\otimes}} I\right)\delta_i(a_i)\left(\overset{i}{\underset{k=1}{\otimes}} I \otimes \overset{n}{\underset{k=i+1}{\otimes}} a_k\right) + \left(\overset{n-1}{\underset{k=1}{\otimes}} a_k \otimes I\right)\delta_n(a_n)$$

is a derivation of $\overset{n}{\underset{i=1}{\otimes}} A_i$. In this case we say that δ_i 's are the $i\ th$ components of d.

Note:

We can say that δ_i , i =2,3 , …..n are compatible with δ_1 if the above condition is satisfied.

Example 2.4

Let $c \in \overset{n}{\underset{i=1}{\otimes}} A_i$ and $\forall a_i \in A_i$ let

$$\delta_i(a_i) = \begin{cases} \delta_c\left(a_1 \otimes \overset{n}{\underset{k=2}{\otimes}} I\right) & i = 1 \\ \delta_c\left(\overset{i-1}{\underset{k=1}{\otimes}} I \otimes a_i \otimes \overset{n}{\underset{k=I+1}{\otimes}} I\right) & 2 \le i < n \\ \delta_c\left(\overset{n-1}{\underset{k=1}{\otimes}} I \otimes a_n\right) & i = n. \end{cases}$$

Then δ_i 's are compatible $i = 1, 2, \ldots, n$. Therefore we have,

$$d\left(\overset{n}{\underset{i=1}{\otimes}} a_i\right) = \delta_c\left(\overset{n}{\underset{i=1}{\otimes}} a_i\right).$$

Example 2.5

Let ξ_i be derivations on A_i , $i = 1, 2, \ldots, n$, then

$$\delta_1(a_1) = \xi_1(a_1) \otimes \overset{n}{\underset{i=2}{\otimes}} I ,$$

$$\delta_i(a_i) = \overset{i-1}{\underset{k=1}{\otimes}} I \otimes \xi_i(a_i) \otimes \overset{n}{\underset{k=i+1}{\otimes}} I \qquad , i = 2, \ldots, n-1$$

and, $\delta_n(a_n) = \overset{n-1}{\underset{k=1}{\otimes}} I \otimes \xi_n(a_n)$, are compatible for,

$$d = \left(\xi_1 \otimes \overset{n}{\underset{k=2}{\otimes}} I\right) + \sum_{i=2}^{n-1}\left(\overset{i-1}{\underset{k=1}{\otimes}} I \otimes \xi_i \otimes \overset{n}{\underset{k=i+1}{\otimes}} I\right) + \left(\overset{n-1}{\underset{k=1}{\otimes}} I \otimes \xi_n\right)$$

see [7].

3. Main Results

We will be in need to the following Proposition which gives a necessary and sufficient condition for some A_i - derivations to be compatible.

Proposition 3.1

For $1 \le i \le n$, δ_i 's are compatible if and only if

$$\delta_{I\otimes\otimes_{k=2}^{n}a_k}(\delta_1(a_1)) = \delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_2(a_2))\left(\otimes_{k=1}^{2} I\otimes\otimes_{k=3}^{n}a_k\right)+$$
$$+\sum_{i=3}^{n-1}\left(I\otimes\otimes_{k=2}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)\delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_i(a_i))\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)$$
$$+\left(I\otimes\otimes_{k=2}^{n-1}a_k\otimes I\right)\delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_n(a_n)).$$

Proof:-

Let δ_i 's are compatible $i=1,2,...,n.$ then there exists a derivation $d:\otimes_{i=1}^{n}A_i\to\otimes_{i=1}^{n}A_i$,where

$$d\left(\otimes_{i=1}^{n}a_i\right)=\delta_1(a_1)\left(I\otimes\otimes_{k=2}^{n}a_k\right)+\sum_{i=2}^{n-1}\left(\otimes_{k=1}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)\delta_i(a_i)\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)$$
$$+\left(\otimes_{k=1}^{n-1}a_k\otimes I\right)\delta_n(a_n)\ . \qquad (3.1)$$

However,

$$d\left(\otimes_{i=1}^{n}a_i\right)=d\left(\left(I\otimes\otimes_{k=2}^{n}a_k\right)\left(a_1\otimes\otimes_{k=2}^{n}I\right)\right)$$
$$=d\left(I\otimes\otimes_{k=2}^{n}a_k\right)\left(a_1\otimes\otimes_{k=2}^{n}I\right)+\left(I\otimes\otimes_{k=2}^{n}a_k\right)d\left(a_1\otimes\otimes_{k=2}^{n}I\right)$$
$$=\{\ \delta_1(I)\left(I\otimes\otimes_{k=2}^{n}a_k\right)+\left(\otimes_{k=1}^{n}I\right)\delta_2(a_2)\left(\otimes_{k=1}^{2}I\otimes\otimes_{k=3}^{n}a_k\right)$$
$$+$$
$$\sum_{i=3}^{n-1}\left(I\otimes\otimes_{k=2}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)\delta_i(a_i)\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)+\left(I\otimes\otimes_{k=2}^{n-1}a_k\otimes I\right)\delta_n(a_n)\}\left(a_1\otimes\otimes_{k=2}^{n}I\right)$$
$$+$$
$$+\left(I\otimes\otimes_{k=2}^{n}a_k\right)\delta_1(a_1)\left(\otimes_{k=1}^{n}I\right).$$

(3.2)

From (3.1) and (3.2) we get,

$$\left(I\otimes\otimes_{k=2}^{n}a_k\right)(\delta_1(a_1))-(\delta_1(a_1))\left(I\otimes\otimes_{k=2}^{n}a_k\right)=-(\delta_2(a_2))\left(\otimes_{k=1}^{2}I\otimes\otimes_{k=3}^{n}a_k\right)\left(a_1\otimes\otimes_{k=2}^{n}I\right)$$

$$-\sum_{i=3}^{n-1}\left(\left(I\otimes\otimes_{k=2}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)(\delta_i(a_i))\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)\right)\left(a_1\otimes\otimes_{k=2}^{n}I\right)$$

$$+\sum_{i=2}^{n-1}\left(\otimes_{k=1}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)(\delta_i(a_i))\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)-\left(I\otimes\otimes_{k=2}^{n-1}a_k\otimes I\right)(\delta_n(a_n))\left(a_1\otimes\otimes_{k=2}^{n}I\right)$$

$$+\left(\otimes_{k=1}^{n-1}a_k\otimes I\right)\delta_n(a_n).$$

Therefore,

$$\delta_{I\otimes\otimes_{k=2}^{n}a_k}(\delta_1(a_1))=-\delta_2(a_2)\left(\otimes_{k=1}^{2}I\otimes\otimes_{k=3}^{n}a_k\right)\left(a_1\otimes\otimes_{k=2}^{n}I\right)+$$

$$+\left(a_1\otimes\otimes_{k=2}^{n}I\right)(\delta_2(a_2))\left(\otimes_{k=1}^{2}I\otimes\otimes_{k=3}^{n}a_k\right)-\sum_{i=3}^{n-1}\left(I\otimes\otimes_{k=2}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)(\delta_i(a_i))\left(a_1\otimes\otimes_{k=2}^{n}I\right)\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)+$$
$$+\sum_{i=3}^{n-1}\left(\otimes_{k=2}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)\left(a_1\otimes\otimes_{i=2}^{n}I\right)(\delta_i(a_i))\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)-\left(I\otimes\otimes_{k=2}^{n-1}a_k\otimes I\right)(\delta_n(a_n))\left(a_1\otimes\otimes_{k=2}^{n}I\right)$$
$$+\left(I\otimes\otimes_{k=2}^{n-1}a_k\otimes I\right)\left(a_1\otimes\otimes_{k=2}^{n}I\right)(\delta_n(a_n)).$$

Then we have,

$$\delta_{I\otimes\otimes_{k=2}^{n}a_k}(\delta_1(a_1))=\delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_2(a_2))\left(\otimes_{k=1}^{2}I\otimes\otimes_{k=3}^{n}a_k\right)+\sum_{i=3}^{n-1}\left(I\otimes\otimes_{k=2}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)\delta_{\left(a_1\otimes\otimes_{k=2}^{n}I\right)}(\delta_i(a_i))\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)$$
$$+\left(I\otimes\otimes_{k=2}^{n-1}a_k\otimes I\right)(\delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_n(a_n))).$$

On the other hand let,

$$\delta_{I\otimes\otimes_{k=2}^{n}a_k}(\delta_1(a_1))=\delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_2(a_2))\left(\otimes_{k=1}^{2}I\otimes\otimes_{k=3}^{n}a_k\right)+\sum_{i=3}^{n-1}\left(I\otimes\otimes_{k=2}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)\delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_i(a_i))\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)$$
$$+\left(I\otimes\otimes_{k=2}^{n-1}a_k\otimes I\right)\delta_{a_1\otimes\otimes_{k=2}^{n}I}(\delta_n(a_n)).$$

Now we need to show that δ_i 's are compatibles, $i=1,2,...,n$.

That is $d:\otimes_{i=1}^{n}A_i\to\otimes_{i=1}^{n}A_i$ is a derivation where for each $\left(\otimes_{i=1}^{n}a_i\right),\left(\otimes_{i=1}^{n}b_i\right)\in\otimes_{i=1}^{n}A_i$,

$$d\left(\left(\otimes_{i=1}^{n}a_i\right)\left(\otimes_{i=1}^{n}b_i\right)\right)=d\left(\otimes_{i=1}^{n}a_i\right)\left(\otimes_{i=1}^{n}b_i\right)+\left(\otimes_{i=1}^{n}a_i\right)d\left(\otimes_{i=1}^{n}b_i\right)$$

, and

$$d\left(\otimes_{i=1}^{n}a_i\right)=\delta_1(a_1)\left(I\otimes\otimes_{k=2}^{n}a_k\right)+\sum_{i=2}^{n-1}\left(\otimes_{k=1}^{i-1}a_k\otimes\otimes_{k=i}^{n}I\right)\delta_i(a_i)\left(\otimes_{k=1}^{i}I\otimes\otimes_{k=i+1}^{n}a_k\right)$$
$$+\left(\otimes_{k=1}^{n-1}a_k\otimes I\right)\delta_n(a_n).$$

Since,

$$d\left(\left(\bigotimes_{i=1}^{n} a_i\right)\left(\bigotimes_{i=1}^{n} b_i\right)\right)=d\left(\bigotimes_{i=1}^{n} a_i b_i\right)=\delta_1\left(a_1 b_1\right)\left(I\otimes\bigotimes_{k=2}^{n} a_k b_k\right)+\sum_{i=3}^{n-2}\left(b_1\otimes\bigotimes_{k=2}^{i-1} a_k b_k\otimes\bigotimes_{k=i}^{n} I\right)\delta_i\left(a_i b_i\right)\left(\bigotimes_{k=1}^{i} I\otimes\bigotimes_{k=i+1}^{n-1} a_k b_k\otimes a_n\right)+$$

$$+\sum_{i=2}^{n-1}\left(\bigotimes_{k=1}^{i-1} a_k b_k\otimes\bigotimes_{k=i}^{n} I\right)\left(\delta_i\left(a_i b_i\right)\right)\left(\bigotimes_{k=1}^{i} I\otimes\bigotimes_{k=i+1}^{n} a_k b_k\right)+\left(\bigotimes_{k=1}^{n-1} a_k b_k\otimes I\right)\left(\delta_n\left(a_n b_n\right)\right)$$

$$=(\delta_1(a_1))\left(I\otimes\bigotimes_{k=2}^{n} a_k\right)\left(\bigotimes_{i=1}^{n} b_i\right)+\left(a_1\otimes\bigotimes_{k=2}^{n} I\right)(\delta_1(b_1))\left(I\otimes\bigotimes_{k=2}^{n} a_k b_k\right)$$

$$+\sum_{i=2}^{n-1}\left(\bigotimes_{k=1}^{i-1} a_k b_k\otimes\bigotimes_{k=i}^{n} I\right)\left(\delta_i\left(a_i b_i\right)\right)\left(\bigotimes_{k=1}^{i} I\otimes\bigotimes_{k=i+1}^{n} a_k b_k\right)$$

$$+\left(\bigotimes_{k=1}^{n-1} a_k b_k\otimes I\right)\left(\delta_n\left(a_n\right)\right)\left(\bigotimes_{k=1}^{n-1} I\otimes b_n\right)+\left(\bigotimes_{i=1}^{n} a_i\right)\left(\bigotimes_{k=1}^{n-1} b_k\otimes I\right)\left(\delta_n\left(b_n\right)\right)$$

$$=\delta_1\left(a_1\right)\left(I\otimes\bigotimes_{k=2}^{n} a_k\right)\left(\bigotimes_{i=1}^{n} b_i\right)+\left(a_1\otimes\bigotimes_{k=2}^{n} I\right)$$

$$\left\{\delta_1(b_1)\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes a_n\right)+\left(b_1\otimes\bigotimes_{k=2}^{n} I\right)\left(\delta_2\left(a_2 b_2\right)\right)\left(\otimes_{k=1}^{2} I\otimes\bigotimes_{k=3}^{n-1} a_k b_k\otimes a_n\right)\right.$$

$$+\sum_{i=3}^{n-2}\left(b_1\otimes\bigotimes_{k=2}^{i-1} a_k b_k\otimes\bigotimes_{k=i}^{n} I\right)\left(\delta_i\left(a_i b_i\right)\right)\left(\bigotimes_{k=1}^{i} I\otimes\bigotimes_{k=i+1}^{n-1} a_k b_k\otimes a_n\right)$$

$$\left.\left(b_1\otimes\otimes_{k=2}^{n-2} a_k b_k\otimes\otimes_{k=n-1}^{n} I\right)\delta_n\left(a_{n-1} b_{n-1}\right)\left(\otimes_{k=1}^{n-1} I\otimes a_n\right)+\left(b_1\otimes\otimes_{k=2}^{n-1} a_k b_k\otimes I\right)\delta_n\left(a_n\right)\right\}$$

$$\left(\bigotimes_{k=1}^{n-1} I\otimes b_n\right)+\left(\bigotimes_{i=1}^{n} a_i\right)\left(\bigotimes_{k=1}^{n-1} b_k\otimes I\right)\left(\delta_n\left(b_n\right)\right)$$

(3.3)

Since,

$$\delta_{I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes a_n}\left(\delta_1\left(b_1\right)\right)=\delta_{b_1\otimes\bigotimes_{k=2}^{n} I}\left(\delta_2\left(a_2 b_2\right)\right)\left(\bigotimes_{k=1}^{2} I\otimes\bigotimes_{k=3}^{n-1} a_k b_k\otimes a_n\right)+$$

$$+\sum_{i=3}^{n-2}\left(I\otimes\bigotimes_{k=2}^{i-1} a_k b_k\otimes\bigotimes_{k=i}^{n} I\right)\delta_{b_1\otimes\bigotimes_{k=2}^{n} I}\left(\delta_i\left(a_i b_i\right)\right)\left(\bigotimes_{k=1}^{i} I\otimes\bigotimes_{k=i+1}^{n-1} a_k b_k\otimes a_n\right)$$

$$\left(I\otimes\otimes_{k=2}^{n-2} a_k b_k\otimes\otimes_{k=n-1}^{n} I\right)\delta_{b_1\otimes\otimes_{k=2}^{n} I}\left(\delta_n\left(a_{n-1} b_{n-1}\right)\right)\left(\otimes_{k=1}^{n-1} I\otimes a_n\right)+\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes I\right)\delta_{b_1\otimes\bigotimes_{k=2}^{n} I}\left(\delta_n\left(a_n\right)\right).$$

Therefore,

$$\delta_1(b_1)\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes a_n\right)+\left(b_1\otimes\bigotimes_{k=2}^{n} I\right)\left(\delta_2\left(a_2 b_2\right)\right)\left(\bigotimes_{k=1}^{2} I\otimes\bigotimes_{k=3}^{n-1} a_k b_k\otimes a_n\right)$$

$$\left(b_1\otimes\otimes_{k=2}^{n-2} a_k b_k\otimes\otimes_{k=n-1}^{n} I\right)\left(\delta_n\left(a_{n-1} b_{n-1}\right)\right)\left(\otimes_{k=1}^{n-1} I\otimes a_n\right)+\left(b_1\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes I\right)\left(\delta_n\left(a_n\right)\right)$$

$$=\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes a_n\right)\left(\delta_1\left(b_1\right)\right)$$

$$+\left(\delta_2\left(a_2 b_2\right)\right)\left(b_1\otimes I\otimes\bigotimes_{k=3}^{n-1} a_k b_k\otimes a_n\right)$$

$$+\sum_{i=3}^{n-2}\left(I\otimes\bigotimes_{k=2}^{i-1} a_k b_k\otimes\bigotimes_{k=i}^{n} I\right)\left(\delta_i\left(a_i b_i\right)\right)\left(b_1\otimes\bigotimes_{k=2}^{i} I\otimes\bigotimes_{k=i+1}^{n-1} a_k b_k\otimes a_n\right)$$

$$\left(I\otimes\otimes_{k=2}^{n-2} a_k b_k\otimes\otimes_{k=n-1}^{n} I\right)\left(\delta_n\left(a_{n-1} b_{n-1}\right)\right)\left(I\otimes\otimes_{k=2}^{n-1} I\otimes a_n\right)+$$

$$\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes I\right)\left(\delta_n\left(a_n\right)\right)\left(b_1\otimes\bigotimes_{k=2}^{n} I\right).$$

(3.4)

By (3.4) and (3.3) we have,

$$d\left(\left(\bigotimes_{i=1}^{n} a_i\right)\left(\bigotimes_{i=1}^{n} b_i\right)\right)=\delta_1\left(a_1\right)\left(I\otimes\bigotimes_{k=2}^{n} a_k\right)\left(\bigotimes_{i=1}^{n} b_i\right)+\left(a_1\otimes\bigotimes_{k=2}^{n} I\right)$$

$$\left\{\left[\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes a_n\right)\delta_1\left(b_1\right)+\delta_2\left(a_2 b_2\right)\left(b_1\otimes I\otimes\bigotimes_{k=3}^{n-1} a_k b_k\otimes a_n\right)\right.\right.$$

$$+\sum_{i=3}^{n-2}\left(I\otimes\bigotimes_{k=2}^{i-1} a_k b_k\otimes\bigotimes_{k=i}^{n} I\right)\left(\delta_i\left(a_i b_i\right)\right)\left(b_1\otimes\bigotimes_{k=2}^{i} I\otimes\bigotimes_{k=i+1}^{n-1} a_k b_k\otimes a_n\right)$$

$$+\left(I\otimes\otimes_{k=2}^{n-2} a_k b_k\otimes\otimes_{k=n-1}^{n} I\right)\left(\delta_n\left(a_{n-1} b_{n-1}\right)\right)\left(b_1\otimes\otimes_{k=2}^{n-1} I\otimes a_n\right)$$

$$\left.\left.+\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes I\right)\left(\delta_n\left(a_n\right)\right)\left(b_1\otimes\bigotimes_{k=2}^{n} I\right)\right]\right\}\left(\bigotimes_{k=1}^{n-1} I\otimes b_n\right)$$

$$+\left(\bigotimes_{i=1}^{n} a_i\right)\left(\bigotimes_{k=1}^{n-1} b_k\otimes I\right)\left(\delta_n\left(b_n\right)\right)=(\delta_1(a_1))\left(I\otimes\bigotimes_{k=2}^{n} a_k\right)\left(\bigotimes_{i=1}^{n} b_i\right)+$$

$$\left(a_1\otimes\bigotimes_{k=2}^{n} I\right)$$

$$\left\{\left[\left(I\otimes\bigotimes_{k=2}^{n-1} a_k b_k\otimes a_n\right)(\delta_1(b_1))+(\delta_2(a_2))\left(\otimes_{k=1}^{2} b_k\otimes\bigotimes_{k=3}^{n-1} a_k b_k\otimes a_n\right)+\right.\right.$$

$$\left(I\otimes a_2\otimes\bigotimes_{k=3}^{n} I\right)\left(\delta_2\left(b_2\right)\right)\left(b_1\otimes I\otimes\bigotimes_{k=3}^{n-1} a_k b_k\otimes a_n\right)+$$

$$\sum_{i=3}^{n-1}\left(\left(I\otimes\bigotimes_{k=2}^{i-2}a_kb_k\otimes\bigotimes_{k=i}^{n}I\right)(\delta_i(a_i))\left(b_1\otimes\bigotimes_{k=2}^{i-1}I\otimes b_i\otimes\bigotimes_{k=i+1}^{n-1}a_kb_k\otimes a_n\right)+\left(I\otimes b_2\otimes\bigotimes_{k=3}^{n-1}a_kb_k\otimes I\right)\delta_n(a_n)\right\}\left(b_1\otimes\bigotimes_{k=2}^{n-1}I\otimes b_n\right)+$$

$$\left(I\otimes\bigotimes_{k=2}^{i-1}a_kb_k\otimes a_i\otimes\bigotimes_{k=i+1}^{n}I\right)(\delta_i(b_i))\left(b_1\otimes\bigotimes_{k=2}^{i}I\otimes\bigotimes_{k=i+1}^{n-1}a_kb_k\otimes a_n\right)\Bigg)+$$

$$\left(I\otimes\otimes_{k=2}^{n-2}a_kb_k\otimes\otimes_{k=n-1}^{n}I\right)(\delta_{n-1}(a_{n-1}))\left(b_1\otimes\otimes_{k=2}^{n-2}I\otimes b_{n-1}\otimes a_n\right)$$

$$+\left(I\otimes\otimes_{k=2}^{n-2}a_kb_k\otimes a_{n-1}\otimes I\right)(\delta_{n-1}(b_{n-1}))\left(b_1\otimes\otimes_{k=2}^{n-1}I\otimes a_n\right)$$

$$+\left(I\bigotimes_{k=2}^{n-1}a_kb_k\otimes I\right)(\delta_n(a_n))\left(b_1\otimes\bigotimes_{k=2}^{n}I\right)\Bigg\}\left(\bigotimes_{k=1}^{n-1}I\otimes b_n\right)+\left(\bigotimes_{i=1}^{n}a_i\right)\left(\bigotimes_{k=1}^{n-1}b_k\otimes I\right)(\delta_n(b_n)).$$

Then we have,

$$d\left(\left(\bigotimes_{i=1}^{n}a_i\right)\left(\bigotimes_{i=1}^{n}b_i\right)\right)=$$

$$\left\{\delta_1(a_1)\left(I\otimes\bigotimes_{k=2}^{n}a_k\right)+\left(a_1\otimes\bigotimes_{k=2}^{n}I\right)(\delta_2(a_2))\left(\bigotimes_{k=1}^{2}I\otimes\bigotimes_{k=3}^{n}a_k\right)\right\}$$

$$\left(\bigotimes_{i=1}^{n}b_i\right)$$

$$+\left(\bigotimes_{i=1}^{n}a_i\right)\left(I\otimes\bigotimes_{k=2}^{n-1}b_k\otimes I\right)(\delta_1(b_1))\left(\bigotimes_{k=1}^{n-1}I\otimes b_n\right)$$

$$+\left(\otimes_{k=1}^{2}a_k\otimes\bigotimes_{k=3}^{n}I\right)$$

$$\left\{(\delta_2(b_2))\left(\bigotimes_{k=1}^{2}I\otimes\bigotimes_{k=3}^{n-1}a_kb_k\otimes a_n\right)+\left(I\otimes b_2\otimes\bigotimes_{k=3}^{n}I\right)(\delta_3(a_3))\right.$$

$$\left(\otimes_{k=1}^{2}I\otimes b_3\bigotimes_{k=4}^{n-1}a_kb_k\otimes a_n\right)+$$

$$\left(I\otimes b_2\otimes a_3\otimes\bigotimes_{k=4}^{n}I\right)(\delta_3(b_3))\left(\bigotimes_{k=1}^{3}I\otimes\bigotimes_{k=4}^{n-1}a_kb_k\otimes a_n\right)+$$

$$\sum_{i=4}^{n-2}\left(\left(I\otimes b_2\otimes\bigotimes_{k=3}^{i-1}a_kb_k\otimes\bigotimes_{k=i}^{n}I\right)(\delta_i(a_i))\left(\bigotimes_{k=1}^{i-1}I\otimes b_i\otimes\bigotimes_{k=i+1}^{n-1}a_kb_k\otimes a_n\right)\right.$$

$$+\left(I\otimes b_2\otimes\bigotimes_{k=3}^{i-1}a_kb_k\otimes a_i\otimes\bigotimes_{k=i+1}^{n}I\right)(\delta_i(b_i))\left(\bigotimes_{k=1}^{i}I\otimes\bigotimes_{k=i+1}^{n-1}a_kb_k\otimes a_n\right)$$

$$+\left(I\otimes b_2\otimes\otimes_{k=3}^{n-2}a_kb_k\otimes\otimes_{k=n-1}^{n}I\right)(\delta_{n-1}(a_{n-1}))\left(\otimes_{k=1}^{n-2}I\otimes b_{n-1}\otimes a_n\right)+$$

$$\left(I\otimes b_2\otimes_{k=3}^{n-2}a_kb_k\otimes a_{n-1}\otimes I\right)(\delta_{n-1}(b_{n-1}))\left(\otimes_{k=1}^{n-1}I\otimes a_n\right)$$

$$\left(\bigotimes_{i=1}^{n}a_i\right)\left(\bigotimes_{k=1}^{n-1}b_k\otimes I\right)(\delta_n(b_n)).$$

Repeating the above process n-1 times, we get,

$$d\left(\left(\bigotimes_{i=1}^{n}a_i\right)\left(\bigotimes_{i=1}^{n}b_i\right)\right)=$$

$$=\left\{(\delta_1(a_1))\left(I\otimes\bigotimes_{k=2}^{n}a_n\right)\right.$$

$$+\sum_{i=2}^{n-1}\left(\bigotimes_{k=1}^{i-1}a_k\otimes\bigotimes_{k=i}^{n}I\right)(\delta_i(a_i))\left(\bigotimes_{k=1}^{i}I\otimes\bigotimes_{k=i+1}^{n}a_k\right)$$

$$\left.+\left(\bigotimes_{k=1}^{n-1}a_k\otimes I\right)(\delta_n(a_n))\right\}\left(\bigotimes_{i=1}^{n}b_i\right)+$$

$$\left(\bigotimes_{i=1}^{n}a_i\right)\left\{\left(I\otimes\otimes_{k=2}^{n-1}b_k\otimes I\right)(\delta_1(b_1))\left(\otimes_{k=1}^{n-1}I\otimes b_n\right)+\sum_{i=2}^{n-2}\left(\bigotimes_{k=1}^{i}I\otimes\bigotimes_{k=i+1}^{n-1}b_k\otimes I\right)(\delta_i(b_i))\left(\bigotimes_{k=1}^{i-1}b_k\otimes\bigotimes_{k=i}^{n-1}I\otimes b_n\right)\right.$$

$$+\left(\bigotimes_{k=1}^{n}I\right)(\delta_{n-1}(b_{n-1}))\left(\bigotimes_{k=1}^{n-2}b_k\otimes I\otimes b_n\right)\Bigg)+\left(\bigotimes_{k=1}^{n-1}b_k\otimes I\right)(\delta_n(b_n))$$

$$\Bigg\}$$

$$=\left(d\left(\otimes_{i=1}a_i\right)\right)\left(\bigotimes_{i=1}^{n}b_i\right)+\left(\bigotimes_{i=1}^{n}a_i\right)d\left(\bigotimes_{i=1}^{n}b_i\right),$$

which implies that δ_i, $i=1,2,....,n$ are compatible, that is, $d:\bigotimes_{i=1}^{n}A_i\to\bigotimes_{i=1}^{n}A_i$ is a derivation where

$$\left(I\otimes\otimes_{k=2}^{n-1}b_k\otimes I\right)\delta_1(b_1)\left(\otimes_{k=1}^{n-1}I\otimes b_n\right)+\sum_{i=2}^{n-2}\left(\bigotimes_{k=1}^{i}I\otimes\bigotimes_{k=i+1}^{n-1}b_k\otimes I\right)\delta_i(b_i)\left(\bigotimes_{k=1}^{i-1}b_k\otimes\bigotimes_{k=i}^{n-1}I\otimes b_n\right)$$

$$+\left(\bigotimes_{k=1}^{n}I\right)\delta_{n-1}(b_{n-1})\left(\bigotimes_{k=1}^{n-2}b_k\otimes I\otimes b_n\right)$$

$$+\left(\bigotimes_{k=1}^{n-1}b_k\otimes I\right)\delta_n(b_n)=d\left(\bigotimes_{i=1}^{n}b_i\right).$$

In the following we introduce the concept of relative compatibility of finite number of A_i - derivations $\forall i=1,2,......,n.$

Definition 3.2

If δ_i *and* δ_i' $,i=2,......,n.$ are compatible with δ_1,then we say that $(\delta_2,\delta_3,.....,\delta_n)$ *is* δ_1-compatible to $(\delta_2', \delta_3', \ldots\ldots, \delta_n')$,written by

$\delta_i \equiv \delta_i' \pmod{\delta_1} \quad \forall i = 2,3,\ldots\ldots,n.$

We express the general form of any element c in the kernel of $\delta_{x \otimes \otimes_{k=2}^{n} I}$ where $c \in \otimes_{i=1}^{n} A_i$ and $x \in A_1$ in terms of some simple tensor product $c = I \otimes \otimes_{k=2}^{n} b_k$, $b \in \otimes_{k=2}^{n} A_k$.

Proposition 3.3

Let A_i be simple $C*-$algebra $\forall i = 1,2,\ldots\ldots,n.$
Let $c \in \otimes_{i=1}^{n} A_i$, for each
$x \in A_1, \quad \delta_{x \otimes \otimes_{k=2}^{n} I}(c) = 0.$ Then
$c = I \otimes \otimes_{k=2}^{n} b_k$ for some
$I \otimes \otimes_{k=2}^{n} b_k \in \otimes_{k=1}^{n} A_k$. Moreover
$\delta_i \equiv \delta_i' \pmod{\delta_1} \quad \forall i = 2,3,\ldots\ldots,n$, if for some
f_i derivations
of A_i , $(\delta_i - \delta_i')(a_i) = \otimes_{k=1}^{i-1} I \otimes f_i(a_i) \otimes \otimes_{k=i+1}^{n} I$, for all
$2 \le i \le n-1$ *and* $, (\delta_n - \delta_n')(a_n) = \otimes_{k=1}^{n-1} I \otimes f_n(a_n).$

Proof.

Firstly, let $c = \sum_{j=1}^{\infty} a_{1_j} \otimes \otimes_{i=2}^{n} b_{i_j}$,where b_{i_j} 's are linearly independent $\forall i = 2,3,\ldots\ldots,n.$ Since

$$\delta_{x \otimes \otimes_{k=2}^{n} I}(c) = \delta_c(x \otimes \otimes_{k=2}^{n} I) = 0.$$

$$thus, \quad c(x \otimes \otimes_{k=2}^{n} I) - (x \otimes \otimes_{k=2}^{n} I)c = 0 \quad \forall x \in A_1,$$

then ,

$$\sum_{j=1}^{\infty}(a_{1_j} \otimes \otimes_{i=2}^{n} b_{i_j})(x \otimes \otimes_{k=2}^{n} I) - (x \otimes \otimes_{k=2}^{n} I)\Big(\sum_{j=1}^{\infty}(a_{1_j} \otimes \otimes_{i=2}^{n} b_{i_j})\Big) = 0.$$

thus, $\sum_{j=1}^{\infty}\Big((a_{1_j} x - x a_{1_j}) \otimes \otimes_{i=2}^{n} b_{i_j}\Big) = 0$.

Secondly, since b_{i_j} 's are linearly independent
$\forall i = 2,3,\ldots\ldots,n,$
Then $\delta_{a_{1_j}}(x) = 0 \qquad \forall x \in A_1$ see [7].

Thus , $a_{1_j} x - x a_{1_j} = 0, \ a_{1_j} \in Z(A_1) = CI$,then $a_{1_j} = \alpha_j I$, $\alpha_j \in C$ see[9].

Finally,

$$c = \sum_{j=1}^{\infty}(a_{1_j} \otimes \otimes_{i=2}^{n} b_{i_j}) = \sum_{j=1}^{\infty}(\alpha_j I \otimes \otimes_{i=2}^{n} b_{i_j}) = \sum_{j=1}^{\infty}(I \otimes (\bar{\alpha}_j \otimes_{i=2}^{n} b_{i_j})).$$

Now let $(\bar{\alpha}_j \otimes_{i=2}^{n} b_{i_j}) = \otimes_{i=2}^{n} b_{i_j}'$. Therefore, $c = (I \otimes \otimes_{i=2}^{n} b_{i_j}')$ the first part of the Proposition is proved.

Let, $(\delta_2, \delta_3, \ldots\ldots, \delta_n)$ *be* $\delta_1 -$ compatible to $(\delta_2', \delta_3', \ldots\ldots, \delta_n')$,then there is a mapping

$$d_i : \bigotimes_{i=1}^{n} A_i \to \bigotimes_{i=1}^{n} A_i \text{ , defined by}$$

$$\forall \otimes_{i=1}^{n} a_i \in \bigotimes_{i=1}^{n} A_i$$

$$d\left(\bigotimes_{i=1}^{n} a_i\right) = \delta_1(a_1)\left(I \otimes \bigotimes_{k=2}^{n} a_k\right) + \sum_{i=2}^{n-1}\left(\bigotimes_{k=1}^{i-1} a_k \otimes \bigotimes_{k=i}^{n} I\right)(\delta_i(a_i))\left(\bigotimes_{k=1}^{i} I \otimes \bigotimes_{k=i+1}^{n} a_k\right) + \left(\bigotimes_{k=1}^{n-1} a_k \otimes I\right)(\delta_n(a_n)),$$

$$= (\delta_1'(a_1))\left(I \otimes \bigotimes_{k=2}^{n} a_k\right) + \sum_{i=2}^{n-1}\left(\bigotimes_{k=1}^{i-1} a_k \otimes \bigotimes_{k=i}^{n} I\right)(\delta_i'(a_i))\left(\bigotimes_{k=1}^{i} I \otimes \bigotimes_{k=i+1}^{n} a_k\right) + \left(\bigotimes_{k=1}^{n-1} a_k \otimes I\right)(\delta_n'(a_n)),$$

is a derivations of $\bigotimes_{i=1}^{n} A_i$.

Moreover,

$$\delta_{I \otimes \bigotimes_{k=2}^{n} a_k}(\delta_1(a_1)) = \delta_{a_1 \otimes \bigotimes_{k=2}^{n} I}(\delta_2(a_2))\left(\bigotimes_{k=1}^{2} I \otimes \bigotimes_{k=3}^{n} a_k\right) +$$

$$+ \sum_{i=3}^{n-1}\left(I \otimes \bigotimes_{k=2}^{i-1} a_k \otimes \bigotimes_{k=i}^{n} I\right)\delta_{a_1 \otimes \bigotimes_{k=2}^{n} I}(\delta_i(a_i))\left(\bigotimes_{k=1}^{i} I \otimes \bigotimes_{k=i+1}^{n} a_k\right)$$

$$+ \left(I \otimes \bigotimes_{k=2}^{n-1} a_k \otimes I\right)\delta_{a_1 \otimes \bigotimes_{k=2}^{n} I}(\delta_n(a_n)).$$

$$= \delta_{a_1 \otimes \bigotimes_{k=2}^{n} I}(\delta_2'(a_2))\left(\bigotimes_{k=1}^{2} I \otimes \bigotimes_{k=3}^{n} a_k\right) +$$

$$+ \sum_{i=3}^{n-1}\left(I \otimes \bigotimes_{k=2}^{i-1} a_k \otimes \bigotimes_{k=i}^{n} I\right)\delta_{a_1 \otimes \bigotimes_{k=2}^{n} I}(\delta_i'(a_i))\left(\bigotimes_{k=1}^{i} I \otimes \bigotimes_{k=i+1}^{n} a_k\right)$$

$$+ \left(I \otimes \bigotimes_{k=2}^{n-1} a_k \otimes I\right)\delta_{a_1 \otimes \bigotimes_{k=2}^{n} I}(\delta_n'(a_n)).$$

Then,

$$\underset{a_1\otimes\overset{n}{\underset{k=2}{\otimes}} I}{\delta}((\delta_2-\delta_2')(a_2))\left(\overset{2}{\underset{k=1}{\otimes}} I\otimes\overset{n}{\underset{k=3}{\otimes}} a_k\right)+$$

$$+\sum_{i=3}^{n-1}\left(I\otimes\overset{i-1}{\underset{k=2}{\otimes}} a_k\otimes\overset{n}{\underset{k=i}{\otimes}} I\right)\underset{a_1\otimes\overset{n}{\underset{k=2}{\otimes}} I}{\delta}((\delta_i-\delta_i')(a_i))\left(\overset{i}{\underset{k=1}{\otimes}} I\otimes\overset{n}{\underset{k=i+1}{\otimes}} a_k\right)$$

$$+\left(I\otimes\overset{n-1}{\underset{k=2}{\otimes}} a_k\otimes I\right)\underset{a_1\otimes\overset{n}{\underset{k=2}{\otimes}} I}{\delta}\left((\delta_n-\delta_n')(a_n)\right)=0.$$

$$Thus,\ \underset{a_1\otimes\overset{n}{\underset{k=2}{\otimes}} I}{\delta}\{\ (\delta_2-\delta_2')(a_2)\left(\overset{2}{\underset{k=1}{\otimes}} I\otimes\overset{n}{\underset{k=3}{\otimes}} a_k\right)+$$

$$\sum_{i=3}^{n-1}\left(I\otimes\overset{i-1}{\underset{k=2}{\otimes}} a_k\otimes\overset{n}{\underset{k=i}{\otimes}} I\right)\left((\delta_i-\delta_i')(a_i)\right)\left(\overset{i}{\underset{k=1}{\otimes}} I\otimes\overset{n}{\underset{k=i+1}{\otimes}} a_k\right)$$

$$+\left(I\otimes\overset{n-1}{\underset{k=2}{\otimes}} a_k\otimes I\right)\left((\delta_n-\delta_n')(a_n)\right)\ \}=0.$$

Using the first part of this Proposition, we have,

$$(\delta_2-\delta_2')(a_2)\left(\overset{2}{\underset{k=1}{\otimes}} I\otimes\overset{n}{\underset{k=3}{\otimes}} a_k\right)+\sum_{i=3}^{n-1}\left(I\otimes\overset{i-1}{\underset{k=2}{\otimes}} a_k\otimes\overset{n}{\underset{k=i}{\otimes}} I\right)\left((\delta_i-\delta_i')(a_i)\right)\left(\overset{i}{\underset{k=1}{\otimes}} I\otimes\overset{n}{\underset{k=i+1}{\otimes}} a_k\right)$$

$$+\left(I\otimes\overset{n-1}{\underset{k=2}{\otimes}} a_k\otimes I\right)\left((\delta_n-\delta_n')(a_n)\right)=I\otimes\otimes_{k=2}^{n} b_{a_k}=d\left(I\otimes\otimes_{k=2}^{n} a_k\right)$$

where d is a derivation on $\otimes_{i=1}^{n} A_i$.

Hence,

$$(\delta_i-\delta_i')(a_i)=\otimes_{k=1}^{i-1} I\otimes f_i(a_i)\otimes\otimes_{k=i+1}^{n} I\ ,$$

for

$$2\le i\le n-1\ \ and,\ \ (\delta_n-\delta_n')(a_n)=\otimes_{k=1}^{n-1} I\otimes f_n(a_n).$$

where, f_i are derivations of A_i $,i=2,3,\ldots\ldots,n.$

The other direction of the Proposition can be proved by using example (2.5), that is, let ξ_i be derivations on A_i , $i=1,2,\ldots,n$, then

$$\delta_1(a_1)=\xi_1(a_1)\otimes\overset{n}{\underset{k=2}{\otimes}} I\ ,$$

$$\delta_i(a_i)=\overset{i-1}{\underset{k=1}{\otimes}} I\otimes\xi_i(a_i)\otimes\overset{n}{\underset{k=i+1}{\otimes}} I\qquad ,i=2,\ldots,n-1$$

and , $\delta_n(a_n)=\overset{n-1}{\underset{k=1}{\otimes}} I\otimes\xi_n(a_n)$,are compatibles , since

$$d=\left(\xi_1\otimes\overset{n}{\underset{k=2}{\otimes}} I\right)+\sum_{i=2}^{n-1}\left(\overset{i-1}{\underset{k=1}{\otimes}} I\otimes\xi_i\otimes\overset{n}{\underset{k=i+1}{\otimes}} I\right)+\left(\overset{n-1}{\underset{k=1}{\otimes}} I\otimes\xi_n\right)$$

.

And then the proof is completed.

Finally, we get a precise form of A_i − derivations (δ_i) $\forall i=1,2,3,\ldots..,n$ in terms of a sequence of derivations $(\xi_{i_j})_{j=1}^{\infty}$ on A_i and their basis $(e_{i_j})_{j=1}^{\infty}$, $\forall i=1,2,3,\ldots..,n$.

Proposition 3.4

Let $\left\{e_{i_j}\right\}_{j=1}^{\infty}$ are bases for A_i $\forall i=1,2,3,\ldots..,n$, and δ_i be A_i − derivations. Then there is sequence $\left\{\zeta_{i_j}\right\}_{j=1}^{\infty}$ of derivations of A_i , such that,

$$\delta_i(a_i)=\begin{cases}\sum_{j=1}^{\infty}\zeta_{1_j}(a_1)\otimes\otimes_{k=2}^{n} e_{k_j} & i=1.\\ \sum_{j=1}^{\infty}(\otimes_{k=1}^{i-1} e_{k_j}\otimes\zeta_{i_j}(a_i)\otimes\otimes_{k=i+1}^{n} e_{k_j}) & 1 \text{ p } i \text{ p } n.\\ \sum_{j=1}^{\infty}(\otimes_{k=1}^{n-1} e_{k_j}\otimes\zeta_{n_j}(a_n)) & i=n.\end{cases}$$

Proof.

Let $a_i, b_i\in A_i$, $\forall i=2,3,\ldots.,n-1.$

Then we have,

$$\delta_i(a_ib_i)=\sum_{j=1}^{\infty}(\otimes_{k=1}^{i-1} e_{k_j}\otimes\zeta_{i_j}(a_ib_i)\otimes\otimes_{k=i+1}^{n} e_{k_j})$$

$$=\delta_i(a_i)(\otimes_{k=1}^{i-1} I\otimes b_i\otimes\otimes_{i+1}^{n} I)+(\otimes_{k=1}^{i-1} I\otimes a_i\otimes\otimes_{i+1}^{n} I)\delta_i(b_i)$$

$$=\sum_{j=1}^{\infty}(\otimes_{k=1}^{i-1} e_{k_j}\otimes\zeta_{i_j}(a_i)b_i\otimes\otimes_{k=i+1}^{n} e_{k_j})+\sum_{j=1}^{\infty}(\otimes_{k=1}^{i-1} e_{k_j}\otimes a_i\zeta_{i_j}(b_i)\otimes\otimes_{k=i+1}^{n} e_{k_j}).$$

Thus,

$$\sum_{j=1}^{\infty}(\otimes_{k=1}^{i-1} e_{k_j}\otimes(\zeta_{i_j}(a_ib_i)-(\zeta_{i_j}(a_i)b_i+a_i\zeta_{i_j}(b_i)))\otimes\otimes_{k=i+1}^{n} e_{k_j})=0.$$

Since, $(e_{i_j})_{j=1}^{\infty}$,are linear independent $\forall i=1,2,\ldots\ldots,n$, see[7].

Therefore, $\zeta_{i_j}(a_ib_i)=(\zeta_{i_j}(a_i)b_i+a_i\zeta_{i_j}(b_i))$.

Then we have $\left\{\zeta_i\right\}_j$ be sequence of derivations on A_i $\forall i=2,\ldots..,n-1.$

Similarly, we can show that $\left\{\zeta_1\right\}_j$ *and* $\left\{\zeta_n\right\}_j$ are sequences of derivations on A_1, A_n respectively.

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