# Chirp Parameter Estimation in Colored Noise Using Cross-Spectral ESPRIT Method

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**Abstract:** In this paper a new method for estimating the parameters of chirp signals (LFM signals) is provided. It is based on an especial quadratic form transform and cross-spectral ESPRIT method. Compared with general approaches, the method here has many prominent virtues such as low complexity, low computational cost and working in relatively low SNR almost without any prior information about coloured noise. The correctness and the validity of the new approach are verified by computer emulations. [Nature and Science. 2005;3(1):75-80].

**Key words:** chirp signals; ESPRIT method; Colored noise; Cross-spectral estimation

## 1 Introduction

Chirp parameter estimation is a well-known problem in signal processing community. Chirp signals occur in many applications, e.g., radar, sonar, bioengineering, gravity waves and seismography. Various spectral analysis techniques have been used to perform chirp signals estimation and detection. Most are based on the maximum likelihood (ML) principle [1]. However, the accuracy of ML strongly depends on the grid resolution in the search procedure. The computational burden may be too high to obtain reasonable accuracy. There are other procedures to this problem. Such as phase unwrapping [2, 3], which is simple but only suitable for estimation of mono-chirp signal under higher signal-to-noise-ratio (SNR) environment; Wigner-Ville distribution (WVD)[4], which is poor in estimation of multi-chirp signals because of Cross-term interferences; Radon transform applied to the Wigner-Ville distribution of the signals (RWD) was suggested in [5], which can be directly extended to the analysis of multi-component chirp signals, but it also has the disadvantage of high complexity.

In this paper a cross-spectral ESPRIT method based on quadratic form transform for detecting and estimating chirp signals is presented. First, using quadratic form transform [6] we can convert nonstationary chirp signals into stationary state. Then the cross-spectral method [7] idea is introduced in ESPRIT [8] to produce a cross-spectral ESPRIT method. Last, the cross-spectral ESPRIT method is applied to process the stationary signal after the quadratic form transform. Replacing the two-dimensional search with mathematical operation, the method in this paper is considerably simpler to implement than ML or RWD. Because of the appliance of the cross-spectral ESPRIT method, it has another advantage that it can restrain independent colored noise and work in relatively low SNR environment.

## 2 Estimation of Frequency Change Rate

### 2.1 Quadratic Form Transform of Chirp Signals

Suppose that the mono-component chirp signal model is:

$$s(t) = A\exp\left\{ j2\pi(f_0 t + \frac{1}{2}mt^2)\right\} \tag{1}$$

Where $A$ denotes the amplitude of chirp signal; $f_0$ denotes initial frequency; $m$ denotes frequency change rate.

Let

$$Z(t) = s(t+\frac{\tau}{2})s^*(t-\frac{\tau}{2}) = A^2\exp\left\{ j2\pi\left(f_0 + mt\right)\tau\right\} \tag{2}$$

It is easy to show that the correlation of $Z$(t) is:

$$R_z(t,\tau^{'}) = E\left[Z(t)Z^*(t-\tau^{'})\right] = A^4\exp\left\{ j2\pi m\tau\tau^{'}\right\} \tag{3}$$

This function is independent of time $t$. In another word, $Z$(t) is a stationary random signal. Hence, via quadratic form transform above, the nonstationary chirp signal is converted into stationary state. So methods for stationary signal processing can be used to do the following treatment. But the transform above is based on mono-chirp signal with no additive noise. In this paper we want to talk about multi-chirp signals and the additive noise is colored. Obviously using the simple transform combined with common stationary signal processing methods (the MUSIC method [9], the ESPRIT method [8] etc.) cannot reach the estimation target. So this paper provides the following cross-spectral ESPRIT method based on an especial quadratic form transform:

Generally in practice, we can acquire only one observed sequence.

$$x(n)=\sum_{i=1}^{q} A_i \exp\left\{ j2\pi(f_i n+\frac{1}{2}m_i n^2)\right\}+\omega_x(n) = s_x(n)+\omega_x(n) \quad (4)$$

With $x(n)$, time delay method [10] is introduced to produce the other three sequences as follows:

$$y(n)=\sum_{i=1}^{q} A_i \exp\left\{ j2\pi[f_i(n+\tau_1)+\frac{1}{2}m_i(n+\tau_1)^2]\right\}+\omega_x(n+\tau_1) = s_y(n)+\omega_y(n) \quad (5)$$

$$z(n)=\sum_{i=1}^{q} A_i \exp\left\{ j2\pi[f_i(n+2\tau_1)+\frac{1}{2}m_i(n+2\tau_1)^2]\right\} +\omega_x(n+2\tau_1) = s_z(n)+\omega_z(n) \quad (6)$$

$$g(n)=\sum_{i=1}^{q} A_i \exp\left\{ j2\pi[f_i(n+3\tau_1)+\frac{1}{2}m_i(n+3\tau_1)^2]\right\} +\omega_x(n+3\tau_1) = s_g(n)+\omega_g(n) \quad (7)$$

Where $A_i(i=1,\cdots,q)$ are amplitudes of chirp signals; $f_i(i=1,\cdots,q)$, $m_i(i=1,\cdots,q)$ are initial frequencies and frequency change rates of chirp signals respectively; $\tau_1$ is a constant, which value is bigger than correlated time of colored noise; $\omega_x(n)$, $\omega_y(n)$, $\omega_z(n)$ and $\omega_g(n)$ are zero-mean independent colored noises with unknown spectral density.

ombining (4)~(7), we use the following especial quadratic form transform:

$$\begin{aligned} x_1(n) &= x(n+\frac{\tau}{2})y^*(n-\frac{\tau}{2}) \\ &= s_x(n+\frac{\tau}{2})s_y^*(n-\frac{\tau}{2})+s_x(n+\frac{\tau}{2})\omega_y^*(n-\frac{\tau}{2}) \\ &+\omega_x(n+\frac{\tau}{2})s_y^*(n-\frac{\tau}{2})+\omega_x(n+\frac{\tau}{2})\omega_y^*(n-\frac{\tau}{2}) \end{aligned} \quad (8)$$

$$\begin{aligned} y_1(n) &= z(n+\frac{\tau}{2})g^*(n-\frac{\tau}{2}) \\ &= s_z(n+\frac{\tau}{2})s_g^*(n-\frac{\tau}{2})+s_z(n+\frac{\tau}{2})\omega_g^*(n-\frac{\tau}{2}) \\ &+\omega_z(n+\frac{\tau}{2})s_g^*(n-\frac{\tau}{2})+\omega_z(n+\frac{\tau}{2})\omega_g^*(n-\frac{\tau}{2}) \end{aligned} \quad (9)$$

Then the correlation of $x_1(n), y_1(n)$ is:

$$r_{x_1y_1}(k)=E[x_1^*(n)y_1(n+k)] = \sum_{i=1}^{q} A_i^4 e^{j4\pi m_i\tau_1(\tau-\tau_1)} e^{j2\pi m_i(\tau-\tau_1)k} \quad (10)$$

For the convenience of notation, let

$\phi_i = 4\pi m_i\tau_1(\tau-\tau_1)$ and $\tau_2=2\pi(\tau-\tau_1)$

Inserting them in (10), we obtain:

$$r_{x_1y_1}(k)=\sum_{i=1}^{q} A_i^4 e^{j\phi_i} e^{jm_i\tau_2 k} \quad (11)$$

## 2.2 Cross-spectral ESPRIT Method

By inserting $r_{x_1y_1}(k)$ in $p\times p$ cross-correlation matrix, we get

$$R_{x_1y_1}=\begin{bmatrix} r_{x_1y_1}(0) & r_{x_1y_1}(-1) & \cdots & r_{x_1y_1}(-p+1) \\ r_{x_1y_1}(1) & r_{x_1y_1}(0) & \cdots & r_{x_1y_1}(-p+2) \\ \vdots & \vdots & \ddots & \vdots \\ r_{x_1y_1}(p-1) & r_{x_1y_1}(p-2) & \cdots & r_{x_1y_1}(0) \end{bmatrix} \quad (12)$$

Then the matrix can be expressed as:

$$R_{x_1y_1}=FE^{j\phi}PF^* \quad (13)$$

Where $F=[F_1,F_2,\cdots,F_q]$ is a $p\times q$ complex matrix, and $F_i=[1,e^{jm_i\tau_2},e^{j2m_i\tau_2},\cdots,e^{j(p-1)m_i\tau_2}]^T$ is a complex column vector; $E^{j\phi}=diag[e^{j\phi_1},e^{j\phi_2},\cdots,e^{j\phi_q}]$ is a complex diagonal matrix; $P=diag[A_1^4,A_2^4,\cdots,A_q^4]$ is a real diagonal matrix.

Let: $r_{x_1'y_1'}(k) = r_{x_1y_1}(k-1) = \sum_{i=1}^{q} A_i^4 e^{j\phi_i} e^{j2\pi m_i(k-1)}$ (14)

Insert $r_{x_1'y_1'}$ in $p \times p$ cross-correlation matrix:

$$R_{x_1'y_1'} = \begin{bmatrix} r_{x_1'y_1'}(0) & r_{x_1'y_1'}(-1) & \cdots & r_{x_1'y_1'}(-p+1) \\ r_{x_1'y_1'}(1) & r_{x_1'y_1'}(0) & \cdots & r_{x_1'y_1'}(-p+2) \\ \vdots & \vdots & \ddots & \vdots \\ r_{x_1'y_1'}(p-1) & r_{x_1'y_1'}(p-2) & \cdots & r_{x_1'y_1'}(0) \end{bmatrix}$$
$$= \begin{bmatrix} r_{x_1y_1}(-1) & r_{x_1y_1}(-2) & \cdots & r_{x_1y_1}(-p) \\ r_{x_1y_1}(0) & r_{x_1y_1}(-1) & \cdots & r_{x_1y_1}(-p+1) \\ \vdots & \vdots & \ddots & \vdots \\ r_{x_1y_1}(p-2) & r_{x_1y_1}(p-3) & \cdots & r_{x_1y_1}(-1) \end{bmatrix} \quad (15)$$

Then we get:

$$R_{x_1'y_1'} = FE^{j\phi}P\Phi^* F^* \quad (16)$$

Where $\Phi = diag[e^{j2\pi m_1}, e^{j2\pi m_2}, \cdots, e^{j2\pi m_q}]$ is a unitary matrix. In the complex field, it is a simple scaling operator.

Theorem: Define $\Gamma$ as the generalized eigenvalue matrix associated with the matrix pencil $\{R_{x_1y_1}, R_{x_1'y_1'}\}$. Then the matrices $\Phi$ and $\Gamma$ are related by

$$\Gamma = \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} \quad (17)$$

to within a permutation of the elements of $\Phi$

Proof: Consider the matrix pencil

$$R_{x_1y_1} - \gamma R_{x_1'y_1'} = FE^{j\phi}P(1-\gamma\Phi^*)F^* \quad (18)$$

When frequency change rates $m_i$ are different, the matrices $F$ and $E^{j\phi}P$ are non-singular evidently. So we get the following equation:

$$rank(R_{x_1y_1} - \gamma R_{x_1'y_1'}) = rank(I - \gamma\Phi^*) \quad (19)$$

$\Phi$ is a $q \times q$ diagonal matrix. So in general

$$rank(R_{x_1y_1} - \gamma R_{x_1'y_1'}) = q \quad (20)$$

However, if

$$\gamma = e^{j2\pi m_i} \quad (21)$$

the $i^{th}$ row of $(I - e^{j2\pi m_i}\Phi^*)$ will become zero. Thus,

$$rank(R_{x_1y_1} - e^{j2\pi m_i} R_{x_1'y_1'}) = rank(I - e^{j2\pi m_i}\Phi^*) = q-1 \quad (22)$$

Consequently, the pencil $\{R_{x_1y_1} - \gamma R_{x_1'y_1'}\}$ will also decrease in rank to $q$-1 whenever $\gamma$ assumes values given by (21). However, by definition these are exactly the generalized eigenvalues (GE's) of the matrix pair $\{R_{x_1y_1}, R_{x_1'y_1'}\}$. Also, since both matrices in the pair span the same subspace, the GE's corresponding to the common null space of the two matrices will be zero, i.e., GE's lie on the unit circle and are equal to the diagonal elements of the rotation matrix $\Phi$, and the remaining $p-q$ GE's are at the origin.

This completes the proof of the theorem.

Once $\Phi$ is known, the estimation of frequency change rates $m_i$ can be obtained. But using the basic cross-spectral ESPRIT method above, the final results are not satisfied because of errors in estimating $R_{x_1y_1}$ and $R_{x_1'y_1'}$ from finite data as well as the morbidity question hiding in the algorithm itself. Herein the TLS-ESPRIT idea [11] is introduced to solve this problem:

The singular value decomposition (SVD) of $R_{x_1y_1}$ is showed as:

$$R_{x_1y_1} = U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V^* \quad (23)$$

Where the columns of $U$ and $V$ are the left and right singular vectors. $\Sigma_1 = diag[\sigma_1, \sigma_2, \cdots, \sigma_q]$ ; Where $\sigma_i (i = 1, \cdots, q)$ is non-zero singular value of $R_{x_1y_1}$, and $\sigma_i \geq \sigma_{i+1} (i = 1, 2, \cdots, q-1)$.

Separate the right singular vector $V$ as $V = [V_1, V_2]$, where $V_1$ is composed of the previous section of $V$ and its rank is $q$; $V_2$ is composed of the follow section which rank is $p$-$q$. In the same way, the left singular vector $U$ is divided into $U = [U_1, U_2]$. From which, we obtain:

$$R_{x_1y_1} = [U_1, U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} \quad (24)$$

Thus $R_{x_1y_1} = U_1 \Sigma_1 V_1^*$

Left multiplied by $U_1^*$ and right multiplied by $V_1$, the expression $R_{x_1y_1} - \gamma R_{x_1'y_1'}$ is equal to $\Sigma_1 - \gamma U_1^* R_{x_1'y_1'} V_1$. So the $p \times p$ generalized eigenvalue problem of the matrix pencil $\{R_{x_1y_1}, R_{\dot{x}_1\dot{y}_1}\}$ turn to be the $q \times q$ generalized eigenvalue problem of the matrix pencil $\{\Sigma_1, U_1^* R_{\dot{x}_1\dot{y}_1} V_1\}$.

Once the generalized eigenvalues $z_i$ of the matrix pencil $\{\Sigma_1, U_1^* R_{\dot{x}_1\dot{y}_1} V_1\}$ are calculated, $m_i$ can be gained from:

$$m_i = \arctan[\mathrm{Im}(z_i) / \mathrm{Re}(z_i)] \qquad (25)$$

It is the cross-spectral ESPRIT estimation of frequency change rates.

## 3 Estimation of Initial Frequencies

Supposing the estimates of $\hat{m}_i (i = 1, \cdots, q)$ are accurate enough, we can consider approximatively $\hat{m}_i \approx m_i$.

Let $r(n) = \sum_{i=1}^{q} e^{-j(2\pi \times \frac{1}{2} \hat{m}_i n^2)}$. (26)

Multiplying it by (4), we get:

$$\begin{aligned} x_2(n) = x(n) r(n) &\approx \sum_{i=1}^{q} A_i \exp\{j 2\pi f_i n\} \\ &+ \omega_x(n) \sum_{i=1}^{q} e^{-j(\pi \hat{m}_i n^2)} \end{aligned} \qquad (27)$$

Applying time delay method also, a sequence $y_2(n)$ that is independent of $x_2(n)$ is produced:

$$\begin{aligned} y_2(n) = x_2(n+\tau_3) &= \sum_{i=1}^{q} A_i \exp\{j 2\pi f_i (n+\tau_3)\} \\ &+ \omega_x(n+\tau_3) \sum_{i=1}^{q} e^{-j\{\pi \hat{m}_i (n+\tau_3)^2\}} \end{aligned} \qquad (28)$$

The correlation of $x_2(n)$, $y_2(n)$ is:

$$\begin{aligned} r_{x_2y_2}(k) &= E[x_2^*(n) y_2(n+k)] \\ &= \sum_{i=1}^{q} A_i^2 e^{j2\pi f_i \tau_3} e^{j2\pi f_i k} \end{aligned} \qquad (29)$$

Using the cross-spectral ESPRIT method depicted in section 2.2, the initial frequency estimates are easily obtained. Herein we do not explain it in detail.

## 4 Simulation

In this section the estimated results of frequency change rates and initial frequencies of chirp signals will be brought forth and we will compare them with outcomes of RWD method.

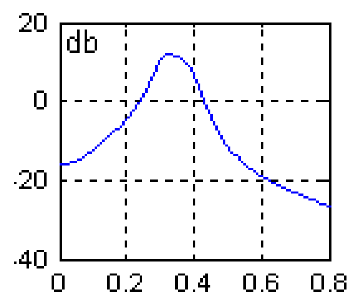
The model of multicomponent chirp signals in colored noise is taken into account as:

$$\begin{aligned} x(n) = A_1 &\exp\left\{ j2\pi (f_1 n + \frac{1}{2} m_1 n^2) \right\} \\ &+ A_2 \exp\left\{ j2\pi (f_2 n + \frac{1}{2} m_2 n^2) \right\} + \omega_x(n) \end{aligned} \qquad (30)$$

Where $m_1 = 0.17$, $m_2 = 0.19$, $f_1 = 0.1$, $f_2 = 0.12$; $\omega_x(n)$ is zero-mean, stationary colored noise with unknown spectral density. It is derived by a white noise with zero mean and variance 1 passing through a band-pass filter, which has the follow expression:

$$H(z) = \frac{k(1 - 2z^{-2} + z^{-4})}{1 - 1.637z^{-1} + 2.237z^{-2} - 1.307z^{-3} + 0.641z^{-4}} \qquad (31)$$

Curve of power spectral density is showed in Figure 1.




Unitary frequency

**Figure 1. Unitary Power Spectral of Colored Noise**

It is easy to obtain the correlation time of colored noise $\omega_x(n)$ be $\tau_0 = 25$. We assume delay time $\tau = 30$.

Let

$y(n) = x(n+30)$, $z(n) = x(n+60)$, $g(n) = x(n+90)$.

As we see, the colored noise in $x(n), y(n), z(n), g(n)$ is independent of each other. Let every data lengths of the four sequences be 512. Both SNRs of two chirp components are –5dB.

After 30 Monte-Carlo simulations under the same test conditions, the statistics of chirp parameter estimates using cross-spectral ESPRIT method are shown in Table 1.

For the convenience of compare, keep the emulational model and conditions of previous test invariable, but $\omega_x(n)$ in model is changed into white Gaussian noise. Applying the well-known Radon-Wigner distribution method, the estimated curve is shown in Figure 2 and the statistics of estimation results are shown in Table 2.

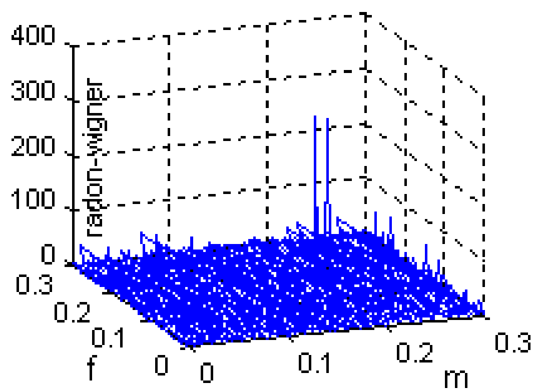



**Figure2 RWD of chirp signals**

We can get from the simulation results that when both SNRs are –5dB, for frequency change rates, the estimated accuracy of cross-spectral ESPRIT method is close to the accuracy of RWD method, but the computational burden of the first method is lower than the second method to heavens; for initial frequency $f$, the accuracy of method in this paper is bad comparing with RWD because of the assumption that $\hat{m}_i \approx m_i$. However, in practice it is often the case that the frequency change rates are the only parameters for interest, so the method in this paper is applicable in engineering. If the high estimated accuracy of parameter $f$ is requested by all means, the method in literature [12] can be used.

**Table 1. Statistic of estimates by cross-spectral ESPRIT method (SNR=-5dB)**

| Parameter | $m_1$ | $m_2$ | $f_1$ | $f_2$ |
|---|---|---|---|---|
| Real value | 0.17 | 0.19 | 0.1 | 0.12 |
| Estimated mean | 0.1700 | 0.1900 | 0.1026 | 0.1213 |
| Estimated variance | 3.6667E-09 | 8.3000E-09 | 1.3998E-04 | 1.5405E-04 |

**Table 2. Statistic of estimates by RWD (SNR=-5dB)**

| Parameter | $m_1$ | $m_2$ | $f_1$ | $f_2$ |
|---|---|---|---|---|
| Real value | 0.17 | 0.19 | 0.1 | 0.12 |
| Estimated mean | 0.1701 | 0.1901 | 0.1001 | 0.12 |
| Estimated variance | 9.2689E-09 | 7.6695E-09 | 2.5673E-08 | 3.3686E-08 |

## 5 Conclusion

In this paper a new approach for detecting and estimating chirp signals is presented. Both theoretical evaluations and simulations prove that cross-spectral ESPRIT method decreases a good number of computational complexities because it avoids the two-dimensional search, which RWD method and so forth must confront. Even working in colored noise and relatively low SNR phenomena, the method here is very accurate, highly reliable, and can operate efficiently.

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