Complex Number

From Wikipedia, the free encyclopedia, http://en.wikipedia.org/wiki/Complex_number

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Abstract: For the recent issues of Nature and Science, there are several articles that discussed the numbers. To offer the references to readers on this discussion, we got the information from the free encyclopedia Wikipedia and introduce it here. Briefly, complex numbers are added, subtracted, and multiplied by formally applying the associative, commutative and distributive laws of algebra. The set of complex numbers forms a field which, in contrast to the real numbers, is algebraically closed. In mathematics, the adjective "complex" means that the field of complex numbers is the underlying number field considered, for example complex analysis, complex matrix, complex polynomial and complex Lie algebra. The formally correct definition using pairs of real numbers was given in the 19th century. A complex number can be viewed as a point or a position vector on a two-dimensional Cartesian coordinate system called the complex plane or Argand diagram. The complex number is expressed in this article. [Nature and Science. 2006;4(2):71-78].

Keywords: add; complex number; multiply; subtract

Editor: For the recnet issues of Nature and Science, there are several articles that discussed the numbers. To offer the references to readers on this discussion, we got the information from the free encyclopedia Wikipedia and introduce it here (Wikimedia Foundation, Inc. Wikipedia, 2006).

In mathematics, a **complex number** is an expression of the form $a + bi$, where a and b are real numbers, and i stands for the square root of minus one (−1), which cannot be represented by any real number. For example, $3 + 2i$ is a complex number, where 3 is called the *real part* and 2 the *imaginary part*.

Since a complex number $a + bi$ is uniquely specified by an ordered pair (a, b) of real numbers, the complex numbers are in one-to-one correspondence with points on a plane, called the complex plane.

The set of all complex numbers is usually denoted by **C**, or in blackboard bold by $\mathbb{C}$. It includes the real numbers because every real number can be regarded as complex: $a = a + 0i$.

Complex numbers are added, subtracted, and multiplied by formally applying the associative, commutative and distributive laws of algebra, together with the equation $i^2 = -1$:

$(a + bi) + (c + di) = (a+c) + (b+d)i$

$(a + bi) - (c + di) = (a-c) + (b-d)i$

$(a + bi)(c + di) = ac + bci + adi + bd\, i^2 = (ac-bd) + (bc+ad)i$

Division of complex numbers can also be defined. The set of complex numbers forms a field which, in contrast to the real numbers, is algebraically closed.

In mathematics, the adjective "complex" means that the field of complex numbers is the underlying number field considered, for example complex analysis, complex matrix, complex polynomial and complex Lie algebra.

Definition

Wikibooks Algebra has more about this subject: ***Complex numbers***

The complex number field

Formally, the complex numbers can be defined as ordered pairs of real numbers (a, b) together with the operations:

$$(a,b) + (c,d) = (a+c, b+d)$$

$$(a,b) \cdot (c,d) = (ac - bd, bc + ad).$$

So defined, the complex numbers form a field, the complex number field, denoted by **C**.

We identify the real number a with the complex number (a, 0), and in this way the field of real numbers **R** becomes a subfield of **C**. The imaginary unit i is the complex number (0, 1).

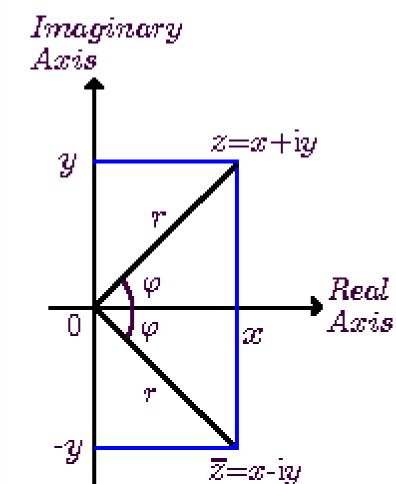
In **C**, we have:

- additive identity ("zero"): (0, 0)
- multiplicative identity ("one"): (1, 0)
- additive inverse of (a,b): ($-a$, $-b$)
- multiplicative inverse (reciprocal) of non-zero (a, b):

$$\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right).$$

C can also be defined as the topological closure of the algebraic numbers or as the algebraic closure of **R**, both of which are described below.

The complex plane



A complex number can be viewed as a point or a position vector on a two-dimensional Cartesian coordinate system called the **complex plane** or **Argand diagram** (named after Jean-Robert Argand).

The Cartesian coordinates of the complex number are the real part x and the imaginary part y, while the circular coordinates are $r = |z|$, called the *absolute value or modulus*, and $\varphi = \arg(z)$, called the *complex argument* of z (mod-arg form). Together with Euler's formula we have

$$z = x + iy = r(\cos\phi + i\sin\phi) = re^{i\phi}.$$

Additionally the notation $r \operatorname{cis} \varphi$ is sometimes used.

Note that the complex argument is unique modulo 2π, that is, if any two values of the complex argument exactly differ by an integer multiple of 2π, they are considered equivalent.

By simple trigonometric identities, we see that

$$r_1e^{i\phi_1} \cdot r_2e^{i\phi_2} = r_1r_2e^{i(\phi_1+\phi_2)}$$

and that

$$\frac{r_1e^{i\phi_1}}{r_2e^{i\phi_2}} = \frac{r_1}{r_2}e^{i(\phi_1-\phi_2)}.$$

Now the addition of two complex numbers is just the vector addition of two vectors, and the multiplication with a fixed complex number can be seen as a simultaneous rotation and stretching.

Multiplication with i corresponds to a counter clockwise rotation by 90 degrees ($\pi / 2$ radians). The geometric content of the equation $i^2 = -1$ is that a sequence of two 90 degree rotations results in a 180 degree (π radians) rotation. Even the fact $(-1) \cdot (-1) = +1$ from arithmetic can be understood geometrically as the combination of two 180 degree turns.

Absolute value, conjugation and distance

The *absolute value* (or *modulus* or *magnitude*) of a complex number $z = r\, e^{i\varphi}$ is defined as $|z| = r$. Algebraically, if $z = a + ib$, then

$$|z| = \sqrt{a^2+b^2}.$$

One can check readily that the absolute value has three important properties:

$$|z| = 0 \text{ iff } z = 0$$
$$|z+w| \leq |z| + |w|$$

for all complex numbers z and w. It then follows, for example, that $|1| = 1$ and $|z/w| = |z|/|w|$. By defining the distance function $d(z, w) = |z - w|$ we turn the complex numbers into a metric space and we can therefore talk about limits and continuity. The addition, subtraction, multiplication and division of complex numbers are then continuous operations. Unless anything else is said, this is always the metric being used on the complex numbers.

The complex conjugate of the complex number $z = a + ib$ is defined to be $a - ib$, written as $\bar{z}$ or z^*. As seen in the figure, $\bar{z}$ is the "reflection" of z about the real axis. The following can be checked:

$$\overline{z+w} = \bar{z} + \bar{w}$$
$$\overline{zw} = \bar{z}\bar{w}$$
$$\overline{(z/w)} = \bar{z}/\bar{w}$$
$$\bar{\bar{z}} = z$$
$$\bar{z} = z \text{ iff } z \text{ is real}$$

$$|z|^2 = z\bar{z}$$
$$z^{-1} = \bar{z}|z|^{-2}$$ if z is non-zero.

The latter formula is the method of choice to compute the inverse of a complex number if it is given in rectangular coordinates.

That conjugation commutes with all the algebraic operations (and many functions; *e.g.*

$\sin \bar{z} = \overline{\sin z}$) is rooted in the ambiguity in choice of i (-1 has two square roots); note, however, that conjugation is not differentiable (see holomorphic).

Complex number division

Given a complex number ($a + ib$) which is to be divided by another complex number ($c + id$) whose magnitude is non-zero, there are two ways to do this; in either case it is the same as multiplying the first by the multiplicative inverse of the second. The first way has already been implied: to convert both complex numbers into exponential form, from which their quotient is easy to derive. The second way is to express the division as a fraction, then to multiply both numerator and denominator by the complex conjugate of the denominator. This causes the denominator to simplify into a real number:

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right).$$

Matrix representation of complex numbers

While usually not useful, alternative representations of complex fields can give some insight into their nature. One particularly elegant representation interprets every complex number as 2×2 matrix with real entries which stretches and rotates the points of the plane. Every such matrix has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with real numbers a and b. The sum and product of two such matrices is again of this form. Every non-zero such matrix is invertible, and its inverse is again of this form. Therefore, the matrices of this form are a field. In fact, this is exactly the field of complex numbers. Every such matrix can be written as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which suggests that we should identify the real number 1 with the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the imaginary unit i with

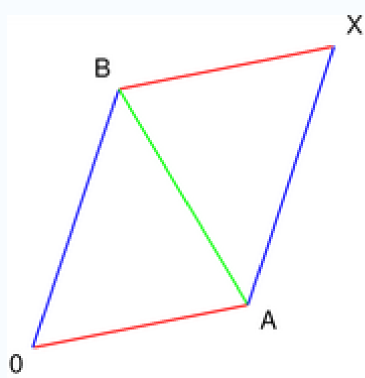
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

a counter-clockwise rotation by 90 degrees. Note that the square of this latter matrix is indeed equal to -1.

The absolute value of a complex number expressed as a matrix is equal to the square root of the determinant of that matrix. If the matrix is viewed as a transformation of a plane, then the transformation rotates points through an angle equal to the argument of the complex number and scales by a factor equal to the complex number's absolute value. The conjugate of the complex number z corresponds to the transformation which rotates through the same angle as z but in the opposite direction, and scales in the same manner as z; this can be described by the transpose of the matrix corresponding to z.

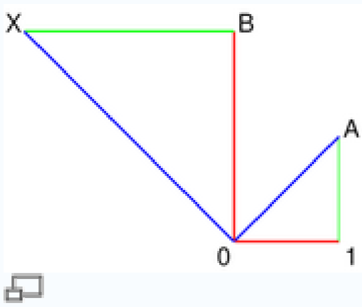
If the matrix elements are themselves complex numbers, then the resulting algebra is that of the quaternions. In this way, the matrix representation can be seen as a way of expressing the Cayley-Dickson construction of algebras.

Geometric interpretation of the operations on complex numbers



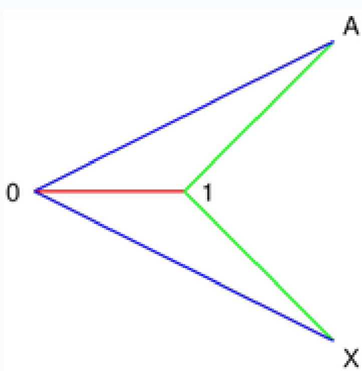
The point X is the sum of A and B.

Choose a point in the plane which will be the origin, 0. Given two points A and B in the plane, their *sum* is the point X in the plane such that the triangles with vertices 0, A, B and X, B, A are similar.



The point X is the product of A and B.

Choose in addition a point in the plane different from zero, which will be the unity, 1. Given two points A and B in the plane, their *product* is the point X in the plane such that the triangles with vertices 0, 1, A, and *0*, B, X are similar.



The point X is the complex conjugate of A.

Given a point A in the plane, its *complex conjugate* is a point X in the plane such that the triangles with vertices 0, 1, A and 0, 1, X are mirror image of each other.

Some properties

Real vector space

C is a two-dimensional real vector space. Unlike the reals, complex numbers cannot be ordered in any way that is compatible with its arithmetic operations: **C** cannot be turned into an ordered field.

R-linear maps **C** → **C** have the general form

$$f(z) = az + b\overline{z}$$

with complex coefficients a and b. Only the first term is **C**-linear; also only the first term is holomorphic; the second term is real-differentiable, but does not satisfy the Cauchy-Riemann equations.

The function

$$f(z) = az$$

corresponds to rotations combined with scaling, while the function

$$f(z) = b\overline{z}$$

corresponds to reflections combined with scaling.

Solutions of polynomial equations

A *root* of the polynomial p is a complex number z such that $p(z) = 0$. A most striking result is that all polynomials of degree n with real or complex coefficients have exactly n complex roots (counting multiple roots according to their multiplicity). This is known as the fundamental theorem of algebra, and shows that the complex numbers are an algebraically closed field.

Indeed, the complex number field is the algebraic closure of the real number field, and Cauchy constructed complex numbers in this way. It can be identified as the quotient ring of the polynomial ring **R**[X] by the ideal generated by the polynomial $X^2 + 1$:

$$\mathbb{C} = \mathbb{R}[X]/(X^2 + 1).$$

This is indeed a field because $X^2 + 1$ is irreducible, hence generating a maximal ideal, in **R**[X]. The image of X in this quotient ring becomes the imaginary unit i.

Algebraic characterization

The field **C** is (up to field isomorphism) characterized by the following three facts:

- its characteristic is 0
- its transcendence degree over the prime field is the cardinality of the continuum
- it is algebraically closed

Consequently, **C** contains many proper subfields which are isomorphic to **C**. Another consequence of this characterization is that the Galois group of **C** over the rational numbers is enormous, with cardinality equal to that of the power set of the continuum.

Characterization as a topological field

As noted above, the algebraic characterization of **C** fails to capture some of its most important properties. These properties, which underpin the foundations of complex analysis, arise from the topology of **C**. The following properties characterize **C** as a topological field:

- **C** is a field.
- **C** contains a subset P of nonzero elements satisfying:
  - P is closed under addition, multiplication and taking inverses.
  - If x and y are distinct elements of P, then either $x-y$ or $y-x$ is in P
  - If S is any nonempty subset of P, then $S+P=x+P$ for some x in **C**.
- **C** has a nontrivial involutive automorphism $x \to x^*$, fixing P and such that xx^* is in P for any nonzero x in **C**.

Given these properties, one can then define a topology on **C** by taking the sets

$$B(x,p) = \{y|p - (y-x)(y-x)^* \in P\}$$

as a base, where x ranges over **C**, and p ranges over P.

To see that these properties characterize **C** as a topological field, one notes that $P \cup \{0\} \cup -P$ is an ordered Dedekind-complete field and thus can be identified with the real numbers **R** by a unique field isomorphism. The last property is easily seen to imply that the Galois group over the real numbers is of order two, completing the characterization.

Pontryagin has shown that the only connected locally compact topological fields are **R** and **C**. This gives another characterization of **C** as a topological field, since **C** can be distinguished from **R** by noting the nonzero complex numbers are connected whereas the nonzero real numbers are not.

Complex analysis

The study of functions of a complex variable is known as complex analysis and has enormous practical use in applied mathematics as well as in other branches of mathematics. Often, the most natural proofs for statements in real analysis or even number theory employ techniques from complex analysis. Unlike real functions which are commonly represented as two dimensional graphs, complex functions have four dimensional graphs and may usefully be illustrated by color coding a three dimensional graph to suggest four dimensions, or by animating the complex function's dynamic transformation of the complex plane.

Applications

Control theory

In control theory, systems are often transformed from the time domain to the frequency domain using the Laplace transform. The system's poles and zeros are then analyzed in the *complex plane*. The root locus, Nyquist plot, and Nichols plot techniques all make use of the complex plane.

In the root locus method, it is especially important whether the poles and zeros are in the left or right half planes, i.e. have real part greater than or less than zero. If a system has poles that are

- in the right half plane, it will be unstable,
- all in the left half plane, it will be stable,
- on the imaginary axis, it will be marginally stable.

If a system has zeros in the right half plane, it is a nonminimum phase system.

Signal analysis

Complex numbers are used in signal analysis and other fields as a convenient description for periodically varying signals. The absolute value $|z|$ is interpreted as the amplitude and the argument $\arg(z)$ as the phase of a sine wave of given frequency.

If Fourier analysis is employed to write a given real-valued signal as a sum of periodic functions, these periodic functions are often written as the real part of complex valued functions of the form

$$f(t) = ze^{i\omega t}$$

where ω represents the angular frequency and the complex number z encodes the phase and amplitude as explained above.

In electrical engineering, the Fourier transform is used to analyze varying voltages and currents. The treatment of resistors, capacitors, and inductors can then be unified by introducing imaginary, frequency-dependent resistances for the latter two and combining all three in a single complex number called the impedance. (Electrical engineers and some physicists use the letter j for the imaginary unit since i is typically reserved for varying currents and may come into conflict with i.) This use is also extended into digital signal processing and digital image processing, which utilize digital versions of Fourier analysis (and Wavelet analysis) to transmit, compress, restore, and otherwise process digital audio signals, still images, and video signals.

Improper integrals

In applied fields, the use of complex analysis is often used to compute certain real-valued improper integrals, by means of complex-valued functions. Several methods exist to do this, see methods of contour integration.

Quantum mechanics

The complex number field is also of utmost importance in quantum mechanics since the underlying theory is built on (infinite dimensional) Hilbert spaces over **C**.

Relativity

In special and general relativity, some formulas for the metric on spacetime become simpler if one takes the time variable to be imaginary.

Applied mathematics

In differential equations, it is common to first find all complex roots r of the characteristic equation of a linear differential equation and then attempt to solve the system in terms of base functions of the form $f(t) = e^{rt}$.

Fluid dynamics

In fluid dynamics, complex functions are used to describe potential flow in 2d.

Fractals

Certain fractals are plotted in the complex plane e.g. Mandelbrot set and Julia set.

History

The earliest fleeting reference to square roots of negative numbers occurred in the work of the Greek mathematician and inventor Heron of Alexandria in the 1st century AD, when he considered the volume of an impossible frustum of a pyramid. They became more prominent when in the 16th century closed formulas for the roots of third and fourth degree polynomials were discovered by Italian mathematicians. It was soon realized that these formulas, even if one was only interested in real solutions, sometimes required the manipulation of square roots of negative numbers. For example, Tartaglia's cubic formula gives the following solution to the equation $x^3 - x = 0$:

$$\frac{1}{\sqrt{3}}\left(\sqrt{-1}^{1/3} + \frac{1}{\sqrt{-1}^{1/3}}\right).$$

At first glance this looks like nonsense. However formal calculations with complex numbers show that the equation $z^3 = i$ has solutions $-i$,

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i \quad \text{and} \quad \frac{-\sqrt{3}}{2} + \frac{1}{2}i.$$

Substituting these in turn for $\sqrt{-1}^{1/3}$ into the cubic formula and simplifying, one gets 0, 1 and -1 as the solutions of $x^3 - x = 0$.

This was doubly unsettling since not even negative numbers were considered to be on firm ground at the time. The term "imaginary" for these quantities was coined by René Descartes in the 17th century and was meant to be derogatory (see imaginary number for a discussion of the "reality" of complex numbers). A further source of confusion was that the equation

$$\sqrt{-1}^2 = \sqrt{-1}\sqrt{-1} = -1$$

seemed to be capriciously inconsistent with the algebraic identity

$$\sqrt{a}\sqrt{b} = \sqrt{ab},$$

which is valid for positive real numbers a and b, and which was also used in complex number calculations with one of a, b positive and the other negative. The incorrect use of this identity (and the related identity

$$\frac{1}{\sqrt{a}} = \sqrt{\frac{1}{a}})$$

in the case when both a and b are negative even bedeviled Euler. This difficulty eventually led to the convention of using the special symbol i in place of

$$\sqrt{-1}$$

to guard against this mistake.

The 18th century saw the labors of Abraham de Moivre and Leonhard Euler. To De Moivre is due (1730) the well-known formula which bears his name, de Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

and to Euler (1748) Euler's formula of complex analysis:

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

The existence of complex numbers was not completely accepted until the geometrical interpretation had been described by Caspar Wessel in 1799; it was rediscovered several years later and popularized by Carl Friedrich Gauss, and as a result the theory of complex numbers received a notable expansion. The idea of the graphic representation of complex numbers had appeared, however, as early as 1685, in Wallis's *De Algebra tractatus*.

Wessel's memoir appeared in the Proceedings of the Copenhagen Academy for 1799, and is exceedingly clear and complete, even in comparison with modern works. He also considers the sphere, and gives a quaternion theory from which he develops a complete spherical trigonometry. In 1804 the Abbé Buée independently came upon the same idea which Wallis had suggested, that $\pm\sqrt{-1}$ should represent a unit line, and its negative, perpendicular to the real axis. Buée's paper was not published until 1806, in which year Jean-Robert Argand also issued a pamphlet on the same subject. It is to Argand's essay that the scientific foundation for the graphic representation of complex numbers is now generally referred. Nevertheless, in 1831 Gauss found the theory quite unknown, and in 1832 published his chief memoir on the subject, thus bringing it prominently before the mathematical world. Mention should also be made of an excellent little treatise by Mourey (1828), in which the foundations for the theory of directional numbers are scientifically laid. The general acceptance of the theory is not a little due to the labors of Augustin Louis Cauchy and Niels Henrik Abel, and especially the latter, who was the first to boldly use complex numbers with a success that is well known.

The common terms used in the theory are chiefly due to the founders. Argand called cosφ + i * sinφ the *direction factor*, and $r = \sqrt{a^2 + b^2}$ the *modulus*; Cauchy (1828) called cosφ + i * sinφ the *reduced form* (l'expression réduite); Gauss used i for $\sqrt{-1}$, introduced the term *complex number* for $a + bi$, and called $a^2 + b^2$ the *norm*.

The expression *direction coefficient*, often used for cosφ + i * sinφ, is due to Hankel (1867), and *absolute value,* for *modulus,* is due to Weierstrass.

Following Cauchy and Gauss have come a number of contributors of high rank, of whom the following may be especially mentioned: Kummer (1844), Leopold Kronecker (1845), Scheffler (1845, 1851, 1880), Bellavitis (1835, 1852), Peacock (1845), and De Morgan (1849). Möbius must also be mentioned for his numerous memoirs on the geometric applications of complex numbers, and Dirichlet for the expansion of the theory to include primes, congruences, reciprocity, etc., as in the case of real numbers.

A complex ring or field is a set of complex numbers which is closed under addition, subtraction, and multiplication. Gauss studied complex numbers of the form $a + bi$, where a and b are integral, or rational (and i is one of the two roots of $x^2 + 1 = 0$). His student, Ferdinand Eisenstein, studied the type $a + b\omega$, where ω is a complex root of $x^3 - 1 = 0$. Other such classes (called cyclotomic fields) of complex numbers are derived from the roots of unity $x^k - 1 = 0$ for higher values of k. This generalization is largely due to Kummer, who also invented ideal numbers, which were expressed as geometrical entities by Felix Klein in 1893. The general theory of fields was created by Évariste Galois, who studied the fields generated by the roots of any polynomial equation

$$F(x) = 0$$

The late writers (from 1884) on the general theory include Weierstrass, Schwarz, Richard Dedekind, Otto Hölder, Berloty, Henri Poincaré, Eduard Study, and Alexander MacFarlane.

The formally correct definition using pairs of real numbers was given in the 19th century.

Appendix (by Editor)

In the end of this ariticle, I am introduce the Wikipedia, the free encyclopedia. It is a stage for all the people in the World to show their opinions/ideas and read other's opinions/ideas freely. Not like most of other publications that claim copurights and keep readers from free using, Wikipedia claims that "the Wikipedia content can be copied, modified, and redistributed *so long as* the new version grants the same freedoms to others and acknowledges the authors of the Wikipedia article used (a direct link back to the article satisfies our author credit requirement)". Is this very benefit to the our human society? Yes, it is!

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