Effect of Ensemble Size on the Spectra of the Deformed Random Matrix Ensemble

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Abstract: Random matrix theory (RMT) models the Hamiltonian of a chaotic system by an ensemble of N-dimensional random matrices, where $N\rightarrow\infty$, conditioned by general symmetry constraints. Some models are introduced to apply RMT to mixed systems. In this paper, we use the deformed random matrices ensemble (DRME) as models of mixed systems, in which the diagonal and off-diagonal matrix elements have different variances. The transition between integrable and chaotic systems is studied numerically by varying the ratio of the variances of the diagonal and off-diagonal matrix elements. Analytical formulae are available for the spacing distributions of 2×2 random-matrix ensembles. We evaluate the spacing distribution $P(s)$ for DRME of different dimensions. We also compare our result with the formula of mixed system that obtained by 2×2 random-matrix models. These formulae agree with the spacing distributions of large chaotic systems but fail for regular systems. On the other hand, the spacing distribution is Gaussian for the case of N=2 while it is exponential for large N. The purpose of this paper is to find, for mixed systems, the effective size of the ensemble where the spectral characteristics converge to those of the infinitely large ones. We show that the convergence of $P(s)$ for mixed systems occurs around $N\approx100$. Naturally, convergence occurs at smaller values of N for more chaotic mixed systems and *vice versa*.

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1. Introduction

Random matrix theory (RMT) provides a framework for describing the statistical properties of spectra for quantum systems whose classical counterpart is chaotic [1]. It models the Hamiltonian of the system by an ensemble of N-dimensional random matrices, subject to some general symmetry constraints. Time-reversal-invariant quantum systems are represented by a Gaussian orthogonal ensemble (GOE) of random matrices when the system has rotational symmetry and by a Gaussian symplectic ensemble otherwise. Chaotic systems without time reversal symmetry are represented by the Gaussian unitary ensemble (GUE). A complete discussion of the level correlations even for these three canonical ensembles is a difficult task. Most of the interesting results are obtained for the limit of $N\rightarrow\infty$. Analytical results have long ago been obtained for the case of N=2 [2]. It yields simple analytical expressions for the nearest-neighbor-spacing (NNS) $P(s)$, renormalized to make the mean spacing equal one. The spacing distribution for the GOE, which is given by [1]

$$P_{GOE}(s)=\frac{\pi}{2}s\,e^{-\frac{\pi}{4}s^2}, \qquad (1)$$

is known as Wigner's surmise. Although the two-dimensional GOE obviously ignores the long range correlations within the spectra, the Wigner surmise provides a surprisingly accurate representation for NNS distributions of large matrices.

Berry and Tabor [3] conjectured that the fluctuations of quantum systems whose classical counterpart is completely integrable are the same as those of an uncorrelated sequence of levels. An infinitely large independent-level sequence can be regarded as a Poisson random process. The NNS distribution is given by

$$P(s)=\exp(-s). \qquad (2)$$

The corresponding result for a two-dimensional ensemble of random diagonal matrices is

$$P(s)=\frac{2}{\pi}\exp\left(-\frac{s^2}{\pi}\right), \qquad (3)$$

which does not agree with that of Eq. (2).

The 2×2 random matrix model has often been used to study the spacing distribution for systems with mixed regular-chaotic dynamics. Most of the investigations on this line assign different distributions for the diagonal and off-diagonal matrix elements. This was done for the first time by Robnik[4] in order to describe the crossover transition from the regular regime to that of the GOE. He used a Poisson form for the distribution of the diagonal element and a Gaussian for the off-diagonal ones. The same approach was followed by several authors, e.g. in [5].The spacing distribution is obtained as

$$P(s,\alpha) = \frac{2sD^2}{\alpha}\exp\left(-\frac{1}{2}\left[\frac{1}{\alpha^2}+1\right]s^2D^2\right)I_0\left(\frac{1}{2}\left[\frac{1}{\alpha^2}-1\right]s^2D^2\right), \qquad (4)$$

where $I_0(.)$ is the modified Bessel function [6]. The mean level spacing is expressed in terms the complete elliptic integral of the second kind [6], $E(.)$,as $D = \frac{1}{\sqrt{\pi}}E(1-\alpha^2)$. The crossover transition between integrability and chaos is modeled by varying the parameter α between 0 and 1.

2. Deformed Random Matrix Ensemble (DRME).

Rosenzweig and Porter [4] were probably the first to formulate a RMT for mixed systems. Their model is governed by a Hamiltonian, which is essentially a sum of two terms, one for the chaotic part of the phase space and one for the regular. It has been elaborated on and further developed by several authors (for reviews, see, e.g. Ref. [5,7]). Ensembles of such Hamiltonian matrices are often referred to as the deformed Gaussian ensembles [8].

We consider a system of many degrees of freedom, some of them chaotic and some regular. The Hamiltonian of the system can be decomposed into two terms,

$$H = (1-\lambda)\, H_{regular} + \lambda\, H_{chaotic}. \qquad (5)$$

The term $H_{regular}$ is a diagonal matrix, and $H_{chaotic}$ will be taken with elements drawn from a random Gaussian ensemble, and the parameter λ is ratios of variances varying from 0 to 1.

Since in this work we are concerned with the statistics intermediate between Poisson and GOE, we will define $H_{regular}$ as the Poissonian ensemble. It will be a diagonal matrix with elements given by $(H_{regular,})_{ij}=E_{0,i}\delta_{ij}$ whose eigenvalues $E_{0,i}$ are independent random variable . The second term, $H_{chaotic}$, will be taken with elements drawn from a random Gaussian ensemble.

3. The Nearest Neighbor Spacing Distribution

Our purpose now is to study the matrix-size effect in DRME, and compare our result with the formula of mixed system that obtained by the corresponding 2×2 random-matrix models. We calculate the NNS distributions of the eigenvalues for ensembles of different dimensions N, varying from $N = 8$ to $N=10^3$ ($\sim \infty$). The number of matrices of each of these ensembles is such that the total number of eigenvalues is the same as in the other ones. Figure 1shows by histograms NNS distributions for ensembles of Hamiltonian matrices with λ = 0, the regular case. The distributions are expected to approach the Poisson distribution as λ~ ∞. The Figure indeed shows a gradual transition of the shape of the NNS distribution from the Gaussian form to the Poissonian as N increases. While we still observe a reasonable agreement between the calculated distribution and the Gaussian shape of Eq.(3) obtained by the 2×2 model(solid line) at $N = 8$ and 10, the agreement deteriorates as N increases. The spectra of the ensembles with $N> 50$ are well described the Poisson distribution (dashed line).

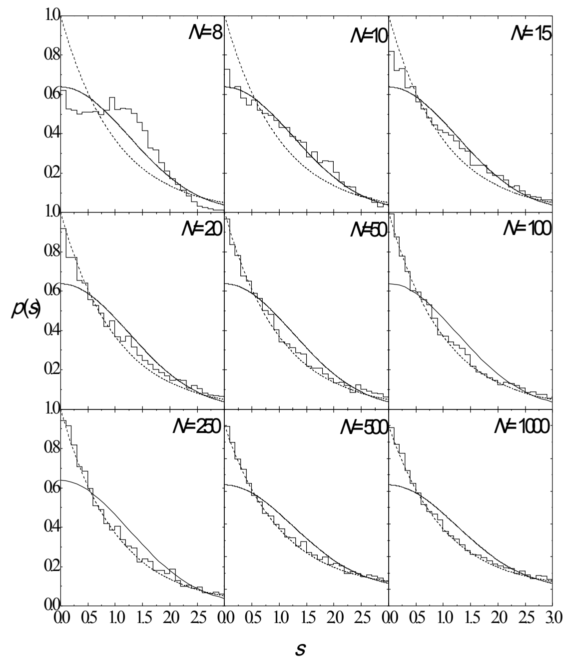

Figure 1 NNS distributions calculated for diagonal RME's (λ = 0 in Eq. (5)) with different dimensions N shown by histograms, with the Gaussian shape of 2×2 represented solid lines, while the dashed ones are calculated using Poisson distribution by Eq. (2)

Figure 2 show the NNS distribution of DRME at λ = 0.001.This distribution may be regarded as a first step of the gradual transition from Poisson statistics to that of GOE. Indeed ,NNS distribution of the deformed 2×2 RME, which is given by Eq. (4) with

$$\alpha = \frac{\sqrt{2}\sigma_{12}}{\sigma} = \sqrt{\frac{2\lambda^2}{1-2\lambda+3\lambda^2}}, \qquad (5)$$

is almost the same as that of the corresponding regular case, which is given by the Gaussian formula (3). As we change the size of matrices, we again observe a gradual transition from the Gaussian shape to the Poissonian as N increases. The transition occurs in the range of $20 \leq N \leq 100$.

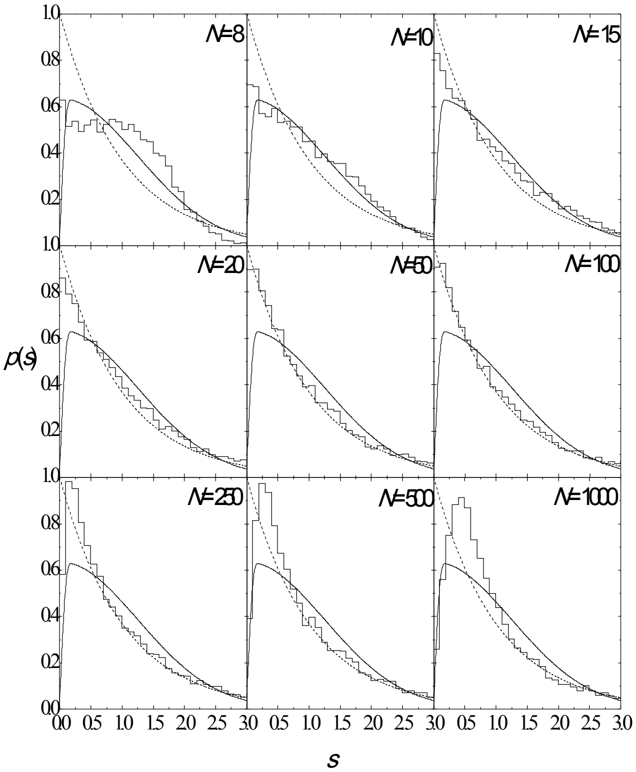

Figure 2 NNS distributions calculated for DRME (λ = 0.001 in Eq. (5)) with different dimensions N shown by histograms, with the Gaussian shape of 2×2 represented solid lines, while the dashed ones are calculated using Poisson distribution by Eq. (2)

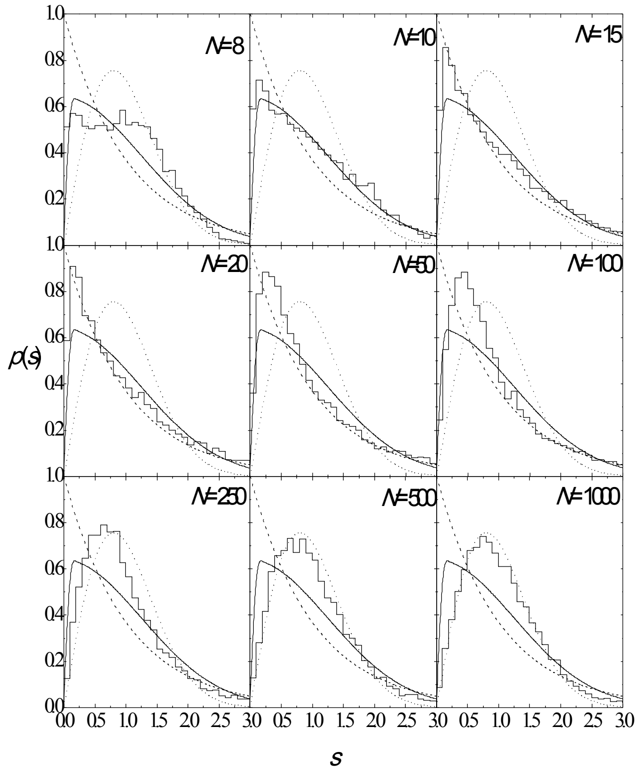

Fig. 3 NNS distributions calculated for DRME (λ = 0.01 in Eq. (5)) with different dimensions N shown by histograms, with the Gaussian shape of 2×2 represented solid lines, while the dashed ones are calculated using Poisson distribution by Eq. (2). The dotted line is calculated using Wigner distribution by Eq. (1).

Figure 3 shows NNS distribution of DEME at $\lambda = 0.01$. The solid curve shows the result for the 2×2 ensemble, given by Eq. (4) with α calculated by Eq. (5).We observe a reasonable agreement with the Gaussian shape of 2×2 at $N = 10$, which gradually deteriorates as the size of ensemble increases. NNS distributions for the large ensembles almost coincide with the Wigner distribution of the GOE.

Figure 4 shows the NNS distributions of DRME of different sizes at $\lambda = 0.1$.In this case the deviation from the formula for the corresponding 2×2 is already observed for the case of$N = 8$.NNS distributions for ensembles with $N \geq 15$ almost agree with the Wigner distribution.

4- Conclusion

In this paper, we study the transition from integrability to chaos for deformed random matrices as a model of mixed system. We consider the transition the effect of dimension N of the ensemble on its NNS distribution.

An analytical result for the corresponding ensemble of 2×2 matrices has previously been obtained. We demonstrate that the Gaussian behavior of the NNS distribution predicted for the ensemble with $N = 2$ case is gradually modified to a Poisson distribution as N increases. We show that NNS distribution of nearly regular mixed system (with λ = 0.001) reasonably agree with the prediction of the 2×2 random-matrix models only for ensembles with $N \leq 20$. We also find the transition to chaos in large DRME occurs for ensembles with $\lambda \geq 0.01$while the distributions of the corresponding 2×2 ensemble are still close to the regular case.

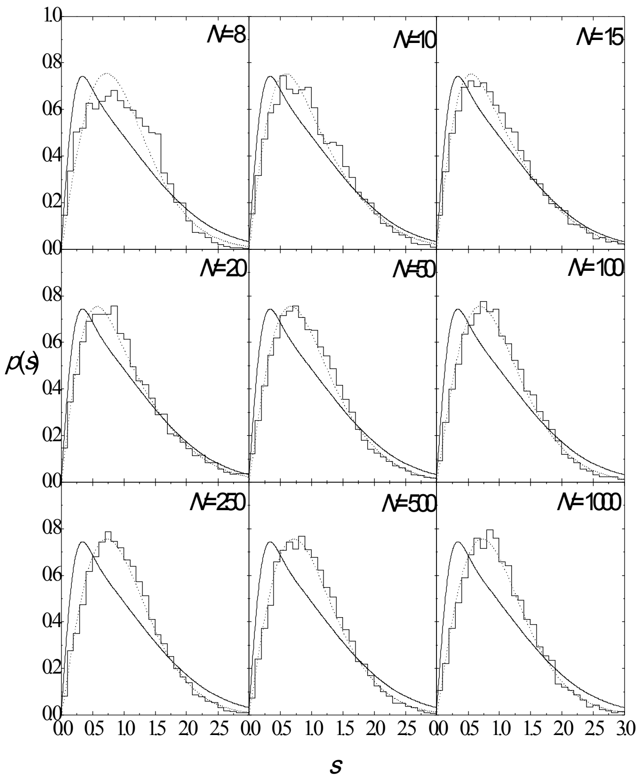

Figure 4 NNS distributions calculated for DRME (λ = 0.1 in Eq. (5)) with different dimensions N shown by histograms, with the Gaussian shape of 2×2 represented solid lines, while The dotted line are calculated using Wigner distribution by Eq. (1).

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