On the Duality of Hardy Spaces $\boldsymbol{H^p}, \mathbf{1} < \boldsymbol{p} < \infty$

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Abstract: We are concerned with the duality of the Hardy spaces of antianalytic functions on the disk is given which generalizes a result of Bukhvalov. So we prove that $H^{pj}(X_j)' \cong \bar{H}^{qj}(X_j')$ under the canonical map when X_j admits analytic projections (p_j). If X be a complex Banach space and $1 < p < \infty, 1/p + 1/q = 1. L^p(X)$ is Lebesgue-Bochner space of X-valued integrable functions on the circle and $\boldsymbol{H^p}(X)$ its Hardy type subspace $\{f \in L^p(X)\ \hat{f}\ (n) = 0\ \forall\ n < 0\}$. Examples are constructed for bad behavior of the analytic projection and of functions in this dual space if X does not belong to the well-known UMD class of Banach spaces.

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1. Introduction

If X be a complex Banach space with dual space $\acute{X}$ and $1 < p < \infty$, $1/p + 1/q = 1$. Let $L^p(X) \coloneqq L^p(\lambda; X)$ be the Lebesgue–Bochner space [15] of X −valued pth power integrable functions on the unit circle T with respect to normalized Lebesgue measure λ on T. The space in the title of this paper is the vector–valued Hardy space. $\boldsymbol{H^p}(X) := \{f \in L^p(X): \hat{f}(n) = 0\ \forall n < 0\}$, where $\hat{f}(n) = (1/2\pi)\int_0^{2\pi} f(e^{it})\, e^{-int} dt$ denotes the nth Fourier coefficient of $f \in (n \in \mathbb{Z})$. As one might expect, $\boldsymbol{H^p}(X)$ can also be realized, via Poisson integral, as a closed subspace of the Hardy space

$$H^p(X) \coloneqq \{f: D \to X \text{ analytic: } \|f\|_p^p :=$$
sup0≤r≤112π02πfreitXpdt<∞

(D denotes the unit disc). (These spaces "coincide" iff X has the "analytic Radon–Nikodým property" a RNP, see 2.7). The analogous spaces of X-valued antianalytic (resp. harmonic) functions are denoted by $\bar{H}^p(X)$ (resp. $h^p(X)$). See Section 2 for details.

It is classical that in the scalar theory ($X = \mathbb{C}$) the dulality $H^{\acute{p}} \cong \bar{H}^q$ holds; more precisely: the canonical map $\bar{H}^q \to H^{\acute{p}}, g \mapsto \left(f \mapsto \int_T\ fgd\lambda\right)$, is an isomorphism [17,7.2]. The crucial ingredient in the proof is the $\|\cdot\|_p$boundedness of the "analytic (or Riesz) projection" $L^p \to H^p$ which assigns to the L^p function $f \sim \sum_{n=-\infty}^{\infty} \hat{f}\,(n) e^{in\vartheta}$ the H^p function $f^a \sim \sum_{n=0}^{\infty} \hat{f}\,(n) e^{in\vartheta}$. Thus, if X is a Banach space such that the analogous map $L^p(X) \to \boldsymbol{H^p}(X), f \mapsto f^a$ is defined and bounded, the isomorphism $\boldsymbol{H^p}(X) \cong \bar{H}^q(\acute{X})$ holds, and the converse is also true (4.5). This was observed first by Bukhvalov [8] and rediscovered in [33]. Unfortunately, the class of Banach spaces admitting this analytic projection is very restrictive: It coincides with the well-known class *UMD* which is, e.g., smaller than the class of superreflexive Bacach space (see 3.3). The question thus arises how to describe $\boldsymbol{H^p}(X)'$ for a general Banach space X. In this paper, $\boldsymbol{H^p}(X)'$ is represented as a certain space $\bar{H}_*^q(X')$ of analytic X'-valued functions on the disc (4.3, 4.4); this space contains $\bar{H}(X')$ as a weak* sequentially dense subspace. In general $\bar{H}^q(X')$ is in $\bar{H}_*^q(X')$ neither dense nor closed (4.7): e.g., sufficient for denseness is the Radon-Nikodym property of X'; in present of a *RNP* it is also necessary.

The connection of this description of $\boldsymbol{H^p}(X)'$ with the analytic projection is still very close: $\bar{H}_*^q(X')$ consist exactly of the antianalytic projections of functions in $h^q(X')$ (Corollary 4.5). (Note that for any harmonic function $\sum_{n=-\infty}^{\infty} a_n\, r^{|n|} e^{in\vartheta}$ the analytic projection $\sum_{n=0}^{\infty} a_n\, z^n$ can always be defined, see 3.1). It must be said, however, that the norm $\|g\|_q^*$ of a member $g \in \bar{H}_*^q(X')$ depends so explicitly on its action as a functional on $\boldsymbol{H^p}(X)$ that the representation theorem can not be regarded as really satisfactory. My justification is, first that the function is $\bar{H}_*^q(X')$—which, anyhow, are the functional on $\boldsymbol{H^p}(X)$—can enjoy a rather unwieldy boundary behavior. Even if X' has the *RNP* (in which case $g \in \bar{H}_*^q(X')$ is the limit in norm of the, $g_r, g_r(z): g(rz), r < 1$, see 4.7). Cf. the discussion in 4.8. This is exemplified by the examples living in c_0, B (predual of JT [46]), l^1. These constructions might be of interest for other vector- valued Hardy space or harmonic analysis as well. Second, the result of Bukhavlov mentioned above subordinates itself naturally under the representation given here (4.5) and, anyway, some assertions about the position of $\bar{H}^q(\acute{X})$ in $\bar{H}^q_*(X') = \boldsymbol{H}^p(X)'$ can be made (4.6, 4.7).

I am not treating the case $p = 1$, since this has been done, for several variants of vector-valued H^1 spaces, by Blasco [3] and Bourgain [5].

The organization of this paper is as follows. In Section 2, we collect preliminaries on vector-valued Hardy spaces. In section 3, the analytic projection operators are introduced, and first example is given of bad behavior of this operation outside the UMD class (3.5). The representation of $\boldsymbol{H}^p(X)'$ described informally above is carried out in Section 4, consider $H^{pj}(X_j)' \cong \bar{H}^{qj}(X_j')$ under the canonical map when X_j admits analytic projections (p_j). Last but not least, the construction of (counter-) examples fills Section 5.

2. Preliminaries on vector-valued Hardy spaces

We begin by recalling some necessary definitions and notation on spaces of integrable functions.

In this paper, D is the unit disc in $\mathbb{C}, T = \partial D$ the unit circle, $\lambda = d\vartheta/2\pi$ normalized Lebesgue measure on T. All spaces of integrable functions will be taken with respect to λ, which is therefore suppressed innotation. χ_E is the indicator function of $E \subset T. X, Y$ denote complex Banach spaces, $X^{\prime}$ the duall space of X, B_X (resp. $\mathring{B}_X$) the closed (resp. open) unit ball of X. The term "isometry" does not include surjectivity, whereas "isomorphism" does. If f is an X-valued and g an $X^{\prime}$-valued function, $\langle f, g\rangle$ stands for the scalar function $\langle f(\cdot), g(\cdot)\rangle. C(T;X)$has it usual meaning and is abbreviated as $C(X); C := C(T;\mathbb{C})$.

The basic theory of the Bochner integral and Bochner-Lebesgue spaces $L^p(X) = L^p(\lambda;X)$ is supposed to be known [15]. We are going to explain the less familiar notion of Gel'fand integral and the spaces $L^p(X^{\prime}, X)$ first. Unless stated otherwise, $1 \le p, q \le \infty$.

Definition 2.1. A function $f: T \to X'$ is called scalarly integrable (w.r.t.X) if the function $\langle x, f\rangle: T \to \mathbb{C}$ is integrable for all $x \in X$ in this case, for any Borel set $E \subset T$, the Gel'fand integral $(G)\int_E f d\lambda \in X'$ is well–defined by the formula

$$\langle x, (G)\int_E f d\lambda\rangle := \int_E \langle x, f\rangle d\lambda, x \in X \text{ [15].}$$

The symbol (G) will often be suppressed.

Now recall that the Banach lattice $L^p(\lambda;\mathbb{R})$ is order complete [28]; i.e., every order–bounded subset of L^p has a supremum in L^p (supremum in the sense of order in L^p, denoted by L^p-sup). Put

$\gamma^p(X^{\prime}, X) = \gamma^p(\lambda; X^{\prime}, X) := \{f: T \to X^{\prime}: \langle x, f\rangle \in L^p \ \forall x \in X\}$
and $\{|\langle x, f\rangle|: \|x\| \le 1\}$ *is order bounded in* L^P.

Following Bukhvalo [8], one defines for $f \in \gamma(X^{\prime}, X)$ the L^p function $v_f := L^P \sup_{\|x\|\le 1}|\langle x, f\rangle|$ and the semi-norm $\|f\|_p: \|v_f\|_p$. The null space of the semi-norm $\|\cdot\|_p$ on $\gamma^p(X^{\prime}, X)$ is easily recognized as $\{f \in \gamma^p(X^{\prime}, X): \forall x \in X: \langle x, f\rangle = 0 \text{ a.e.}\}$. (Note that in the formulation "$\forall x \in X: \langle x, f\rangle = 0$ a.e." the exceptional null set depends on x. Typical example: $f: T \to l^2(T) := e_t$, the tth unit vector. We have, $\|f(\cdot)\| = 1$, but $\|f\|_p = 0$.)

Finally, put $L^p(X^{\prime}, X) = L^p(\lambda; X^{\prime}, X) := \gamma^p(X^{\prime}, X)/\|\|\cdot\|\|_p^{-1}(0)$ with the associated norm $\|\varphi\|_p = \|f\|_p \left(f \in \varphi \epsilon L^p(X^{\prime}, X)\right)$. For $\varphi \epsilon L^p(X^{\prime}, X)$ $\langle x,\varphi\rangle \in L^p(x\in X)$, and $(G)\int_E \varphi d\lambda \ (E \subset T \ Borel)$ are well-defined, since independent of the choice of representative. Obviously, we have $L^p(X^{\prime}) \subset L^p(\lambda; X^{\prime}, X)$ as a closed subspace (i.e., the canonical map is an isometry.

Now let $1/p + 1/q = 1, f \in L^p(X), g \in L^q(X^{\prime}, X)$. It is not hard to see that $\langle f, g\rangle: T \to \mathbb{C}$ is a (well-defined!) member of L^1 satisfying $\int_T \langle f, g\rangle d\lambda \le \|f\|_p \|g\|_q$,
so that g acts as a bounded linear functional of norm at most $\|g\|_q$, on $L^p(X)$. The importance of the space $L^p(X^{\prime}, X)$lies in the fact that it is the exact dual space of $L^p(X) (1 \le p < \infty)$:

Theorem 2.2. Let X be a Banch space, $1 \le p < \infty, 1/p + 1/q = 1$ the map

$$L^q(X^{\prime}, X) \to L^p(X)', g \mapsto \int_T \langle\cdot, g\rangle d\lambda =: \langle\cdot, g\rangle$$

is a (well-defined) isometric isomorphism.

It is in a canonical sense equivalent [21] to the perhaps more widespread representation of $L^p(X)'$ using the upper integral [24].

Lemma 2.3. For $\varphi \epsilon L^p(X^{\prime}, X)$, we denote by $\hat{\varphi}(n) = \int_0^{2\pi} e^{-\text{int}} \varphi(t) dt/2\pi \in X^{\prime}$ the nth Fourier coefficient $(n \in \mathbb{Z})$ and by $P[\varphi]: D \to X'$, $P[\varphi](re^{i\vartheta}) := \int_0^{2\pi} P_r(\vartheta - t)\varphi(t) dt/2\pi$ the Poission integral of φ. (The integrals, of course, Gel'fand integrals.) Here P_r is the Poisson kernel,

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}.$$

An easy computation yields, as in the scalar case,

$$P[\varphi](re^{i\vartheta}) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) r^{|n|} e^{in\vartheta}$$

with absolutely and in D locally uniformly convergent series.

Now let $Y \subset X'$ be a Banach subspace. The following conditions on $\varphi \in L^p(X^{'}, X)$ are equivalent [21] :

$\int_E \varphi d\lambda \in Y \forall E \subset T$ Borel;

$\int_T g\varphi d\lambda \in Y \quad \forall g \in L^p (1/p + 1/q = 1)$;

$\hat{\varphi}(n) \in Y \quad \forall n \in \mathbb{Z}$; $P[\varphi](z) \in Y \quad \forall z \in D$.

The space of those $\varphi \epsilon L^p(X^{'}, X)$ is denoted by $L_Y^p(X^{'}, X)$ Obviously $L^p(y) \subset L^p(X', X) \subset L^p(X', X)$ as closed subspaces.

An important feature of functions in $L^p(Y)$ –that is, of the strongly measurable members of $L_Y^p(X^{'}, X)$ –is exhibited in the following.

Theorem 2.4. Let $f \in L^p(y)(1 \le P < \infty)$ $(resp. f \in C(Y))$. Then $P_r * f \to f (r \to 1)$ in $L^p(Y)\ (resp. in\ C(Y))$.

Proof. Since translation in the argument of a function $f \in L^p(Y)\ (resp. C(Y))$ is a continuous map $T \to L^p(Y)$ [23] (here the Bochner integrability enters), the scalar proof can be carried over without difficulty [8].

Corollary 2.5. Trigonometric polynomials $\sum_{n=-N}^{N} y_n e^{in\vartheta}$ $(y_n \in Y)$ are dense in $L^p(Y), 1 \le p < \infty$ and in $C(Y)$.

Proposition 2.6. For $\varphi \epsilon L^p(X^{'}, X)$ we denote by $C[\varphi]: D \to X^{'}$,

$C[\varphi](re^{i\vartheta}) := \int_0^{2\pi} C_r(\vartheta - t)\varphi(t)dt/2\pi$

Cauchy integral of φ. Here C_r is the Cauchy kernel, $C_r(t) = 1/(1 - re^{it}). C[\varphi]$ is an analytic $X^{'}$ –valued function with Taylor series $C[\varphi](z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n$; in particular, $C[\varphi]: D \to Y$ if $\varphi \epsilon L_y^p(X^{'}, X)$, where $Y \subset X'$ is a Banach subspace. Comparing the coefficients of $P[\varphi]$ and $C[\varphi]$ yields: $\hat{\varphi}(n) = 0 \forall n < 0 \Leftrightarrow P[\varphi]$ is analytic $\Leftrightarrow P[\varphi] = C[\varphi]$. Any $\varphi \epsilon L^p(X^{'}, X)$ with these properties is called "of analytic type " ("analytic" for short). The (obviously closed) subspace of analytic members of $L^p(X^{'}, X)$ $(resp. L_Y^p(X^{'}, X))$ $(resp. L^p(Y))$ is donted by $(resp. L_{y,a}^p(X^{'}, X))$ $(resp. L_a^p(Y))$; of course, $L_a^p(Y) \subset L_{y,a}^p(X^{'}, X) \subset L_a^p(X^{'}, X)$.

As a corollary of Theorem 2.4. "analytic" polynomials $\sum_{n=0}^{N} y_n e^{in\vartheta}$ $(y_n \in Y)$ are dense in $L_a^p(Y), 1 \le p < \infty$.

Proposition 2.7. We define $h^p(Y) := \{ u: D \to Y$ harmonic $\|u\|_p < \infty \}$ where $\|u\|_p = \sup_{r<1} \|u_r\|_p$ and $u_r : T \to Y; u_r(e^{it}) := u(re^{it})$. A word on the notion of a Banach space valued harmonic function seems in order. Exactly as in the better known case of holomorphic functions [23] any two reasonable definitions of harmonicity for a Banach space valued function are equivalent. To be more specific, any of the following conditions on $u: D \to Y$ implies all the others [21] .

i. u is strongly harmonic, i.e, $u \epsilon C^2(D, Y)$ and $\Delta u = 0$.

ii. u is weakly harmonic, i.e. $\langle u, y' \rangle$ is harmonic $\forall y' \in Y'$.

iii. (If $Y = X'$) u is weak* harmonic, i.e., $\langle x, u \rangle$ is harmonic $\forall x \in X$.

iv. $\exists y_n \in Y\ (n \in \mathbb{Z})$ such that $u(re^{i\vartheta}) = \sum_{n=-\infty}^{\infty} y_n r^{|n|} e^{in\vartheta}$ in D with absolutey and locally uniformly convergent series.

By the usual sub harmonicity argument it is easily proved that $\|u_r\|_p$ increases with r for $u: D \to Y$ harmonic, $1 \le p \le \infty$ [21]. Also, one has the scale $h^1(Y) \supset h^p(Y) \supset h^q(Y) \supset h^\infty(Y)$ if $1 \le p \le q \le \infty$, the inclusions are of norm ≤ 1.

The following Poisson integral representation theorem [21Theorem (1.5)] is essentially a concise formulation of results of Grossetete [18, Sect. l] and Bukhvalov [8, Theorem 2.3]. Let $Y \sqsubset X'$ as in Lemma 2.2. (e.g., $X = Y'$.

Theorem 2.8. The Poisson integral defines

1^0 an isometry $P: L_Y^1(X', X) \to h^1(Y), \varphi \mapsto P[\varphi]$

2^0 an isometric isomorphism

$P: L_Y^p(X', X) \to h^p(Y), \varphi \mapsto P[\varphi]$ if $1 < p \le \infty$.

For the sake of clarity, I remark that the isometry in 1^0 is never subjective (except $Y = 0$). The full representation space for $h^1(Y)$ would be $M(Y)$, the space of Y –valued $(\sigma - additive)$vector measures with bounded variation on the Borel sets of T. The space $L_Y^p(X^{'}, X)$ appearing above corresponds via the identification $\varphi \mapsto \varphi d\lambda$ exactly to the subspace $M_c(Y)$ consisting of λ-absolutely continuous members of $M(Y)$, this is essentially the "generalized theorem of Lebesgue Nikodym"[16].

Definition 2.9. $\boldsymbol{h}^p(Y) := \{P[\varphi] := \varphi \in L^p[Y], 1 \le p \le \infty\}$. Thus $\boldsymbol{h}^p(Y) \subset h^p(Y)$ is a closed subspace and $\boldsymbol{h}^p(Y) \cong L^p(Y)$ isometrically; for $p \le q$, $\boldsymbol{h}^q(Y) \subset \boldsymbol{h}^p(Y)$ with norm ≤ 1.

Function in these spaces behave well as regards boundary values:

Proposition 2.10. If $u = P[\varphi] \in \boldsymbol{h}^p(Y)$ with $\varphi \in L^p(Y)$ then $\lim_{r\to 1} u(re^{i\vartheta}) = \varphi(\vartheta)$ a.e. Conversely, if $1 < p \le \infty, u \in h^p(Y)$ and $\lim_{r\to 1} u(re^{i\vartheta}) =: u^*(\vartheta)$ exists a.e., then $u^* \in L^p(Y)$ and $u = P[u^*] \in h^p(Y)$.

Summing up,

$\boldsymbol{h}^p(Y) = \{u \in h^p(Y) : u \textit{ has radial limits a.e.}\}$, $1 < p \le \infty$.

Proposition 2.11. The analytic vector–valued Hardy spaces $H^p(Y)$ are defined in the range $0 < p \le \infty$ as

$$\boldsymbol{H}^p(Y) := \left\{ f : D \to Y \textit{ analytic} : \|f\|_p^p := \sup_{r<1} \int_0^{2\pi} \|f(re^{i\vartheta})\|_p^p \frac{d\vartheta}{2\pi} < \infty \right\}$$

$(p < \infty)$,

$$\boldsymbol{H}^\infty(Y) := \left\{ f : D \to Y \textit{ analytic} : \|f\|_\infty = \sup_{z\in D} \|f(z)\| < \infty \right\}.$$

Thus, of course, $H^p(Y) = \{f \in h^p(Y) : f$ is analytic$\}$, $1 < P \le \infty$. We also define the vector–valued Nevanlinna class

$$N(Y) := \left\{ f : D \to Y \textit{ analytic } \|f\|_0 := \sup_{r<1} \exp \int_0^{2\pi} \ln^+ \|f(re^{i\vartheta})\| \frac{d\vartheta}{2\pi} < \infty \right\}$$

Again, the suprema are increasing limits as $r \nearrow 1$ and we have the scale .

$$N(Y) \supset H^p(Y) \supset H^q(Y) \supset h^\infty(Y)$$

if $0 < p \le q \le \infty$;

the first inclusion is because $\|f\|_0^P \le 1 + \|f\|_P^P$ for $f : D \to Y$ analytic (use Jensen's inequality), the other inclusions are clearly of norm ≤ 1. We will make use of the following result due to Danilevich [14] in a more general Frechet space setting. For a simpler proof in the Banach space context [21].

Proposition 2.12. Let X be a separable Banach space and $f \in N(X')$. Then $\lim_{r\to 1} f(re^{i\vartheta})$ exists a.e (ϑ) in $(X', \sigma(X', X))$.

Returning to the range $1 \le P \le \infty$, the following Poisson integral representation theorem is, at least if $p < 1$,a trial consequence of the preceding one (2.7), by the remarks made in 2.6. It is due to Ryan [34]. Let again $Y \subset X'$.

Theorem 2.13. For $1 \le p \le \infty$ the Poisson (or Cauchy) integral defines an isometric isomorphism $p : L^p_{Y,a}(X', X) \longrightarrow H^p(Y), \varphi \mapsto P[\varphi] = C[\varphi]$.

In view of Theorem 2.8 (1°) and the remarks following it, the theorem for $p = 1$ is tantamount to the knowledge that every "analytic" member of $M(Y)$ is already in $Mc(Y)$, i.e., the vector-valued F and M . Riesz theorem [18, 2. Corollary; 21, Theorem (2.3); 25, p. 316; 35, Theorem l] which in turn is a trivial consequence of the scalar-valued one.

Definition 2.14. $\boldsymbol{H}^p(Y) := \{P[\varphi] : \varphi \in L^P_a(Y)\}$ $1 \le P \le \infty$.

Thus $\boldsymbol{H}^p(Y) \subset H^p(Y)$ is closed subspace and for $1 \le p \le q \le \infty, \boldsymbol{H}^q(Y) \subset \boldsymbol{H}^p(Y)$.

As one might expect, the assertions of Preposition 2.7 hold for $\boldsymbol{H}^p(Y)$ in the full range $1 \le p \le \infty$; that is, $\boldsymbol{H}^p(Y) = \{f \in H^p(Y) : f \textit{ has a.e. radial limits}\}$ $1 \le p \le \infty$ [11]. In most of what follows, we will identify the spaces $\boldsymbol{H}^p(Y)$ and $L^p_a(Y)$, more precisel y$\varphi \in L^p_a(Y)$ with $P[\varphi] \in \boldsymbol{H}^p(Y)$ and $\boldsymbol{H}^p(Y)$ $f \in \boldsymbol{H}^p(Y)$ with its boundary value $f^* \in L^p_a(Y)$.

Proposition 2.15. Bukhualov and Danilevich were the first to recognize the close connection between the Radon-Nikodym property (RNP) [15] and the theory of valued h^p spaces. Their result may be summarized as follows: Y has RNP iff $h^p = \boldsymbol{h}^p$ (for one (all) $p \in (1, \infty)$ various extensions of this theorem, as regards the extreme values of p, have been given independently by Blasco [2] and the author [21]. I state here only what is needed later.

Theorem 2.16. Y has the RNP iff $h^\infty = \boldsymbol{h}^\infty$ (that is, by Proposition 2.7, iff every bounded harmonic function $u : D \to Y$ has radial limits $a.e.$).

Theorem 2.17. A Banach space Y has the analytic Radom-Nikodym property if the following equivalent properties are satisfied:

1^o $H^\infty(Y) = \boldsymbol{H}^\infty(Y)$ (that is by Proposition 2.11., every bounded analytic function $f : D \to Y$ has radial limits a.e.).

2^o For all $p \in [1, \infty], H^p(Y) = \boldsymbol{H}^p(Y)$ that is by Proposition 2.11, every $f \in H^p(Y)$ has radial limits a.e.).

3^o Every $f \in N(Y)$ has radial limits a.e.

Proof. It obvioustly suffices to show $1^o \Longrightarrow 3^o$. But this follows trivially from the vector–valued F. and

R. If $f \in N(Y)$, then $f = g/h$ with $g \in H^\infty(Y)$, $h \in H^\infty$ without zeroes.

For example, c_0 does not have a RNP: Consider $f: D \to c_{0, f(z) := (z^n)_{n\in\mathbb{N}}}$. It is also clear from the above that *RNP* implies a *RNP*. The converse is not true; an example is provided by the space L^1 which has a *RNP*, as does every Banach lattice not containing c_0. This major result is again due to Bukhvalov and Danilevich [11], for a simplified proof using semi-embedding).

3. Analytic projection

As in the scalar-valued case, the analytic (or Riesz) projection is intimately connected with the description of duals of Hardy spaces. Let $Y \subset X'$.

Definition 3.1. For a harmonic function $u: D \to Y$ with series $u(re^{i\vartheta}) = \sum_{n=-\infty}^{\infty} y_n r^{|n|} e^{in\vartheta}$, let $u^a: D \to Y$ be the analytic function $u^a(z) = \sum_{n=0}^{\infty} y_n z^n$. u^a is called the analytic projection of u.

Thus, for $\varphi \in L_Y^1(X', X)$, $P[\varphi]^a(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n = C[\varphi](z)$; for simplicity, this will often be abbreviated to $\varphi^a(z)$ there may or may not be a $\psi \in L_Y^1(X', X)$ with $P[\psi] = P[\varphi]^a$ (equivalently, with formal Fourier series $\psi \sim \sum_{n=0}^{\infty} \hat{\varphi}(n) e^{in\vartheta}$); if there is, this (necessarily unique) ψ is also dented by φ^a and called the analytic projection of φ for example, the analytic projection of a trigonometric polynomial $\varphi(e^{i\vartheta}) = \sum_{n=-N}^{N} y_n e^{in\vartheta}$ $(y_n \in Y)$ is

$$\varphi^a(e^{i\vartheta}) = \sum_{n=0}^{N} y_n e^{in\vartheta}.$$

For technical reasons, the antianalytic projection, denoted by ${}^a u$, ${}^a\varphi$, will also be used: For u, φ as above,

$${}^a u(re^{i\vartheta}) = \sum_{n=-\infty}^{0} y_n r^{|n|} e^{in\vartheta} = \sum_{n=0}^{\infty} y_{-n} z^{-n} \; (z = re^{i\vartheta}),$$

${}^a P[\varphi](z) = \sum_{n=-\infty}^{0} \hat{\varphi}(n) z^n = \bar{C}[\varphi] =: {}^a\varphi(z)$, etc. (Here $\bar{C}$ denotes convolution with the complex conjugate of the Cauchy kernel.) It is the "adjoint" of the analytic projection in the sense that, e.g., for trigonometric polynomials

$$\varphi: T \to Y, \psi: T \to Y': \langle \varphi^a, \psi \rangle = \int \langle \varphi, {}^a\psi \rangle d\lambda$$

$$= \int \langle \varphi, {}^a\psi \rangle d\lambda = \langle \varphi, {}^a\psi \rangle \left(= \sum_{n \ge 0} \hat{\varphi}(n) \hat{\psi}(-n)\right).$$

Definition 3.2. For $1 < P < \infty$, we say "Y admits analytic" projection (P)" if $u \mapsto u^a$ is a bounded operator $\boldsymbol{h}^p(Y) \to \boldsymbol{H}^p(Y)$. Equivalent conditions are $:\varphi \mapsto \varphi^a$ is a bounded operator $L^p(Y) \to L_a^p(Y)$ (or, by denseness, only on the trigonometric polynomials); alternatively: $\varphi \mapsto C[\varphi]$ is a bounded operator $L^p(Y) \to \boldsymbol{H}^p(Y)$. One can also show[22] that it is the same to demand that $u \mapsto u^a$ is a bounded operator $h^p(Y) \to H^p(Y)$, or that $H^p(Y)$ is complemented in $\boldsymbol{h}^p(Y)$. By duality (see Definition 3.1), Y admits analytic projection (p) iff Y' admits [anti-] analytic projection (q), $1/p + 1/q = 1$ [8].

Lemma 3.3. That $L^p(Y)$ boundedness of the analytic projection is equivalent to $L^p(\mathbb{R}; Y)$ −boundedness of the Hilbert transform H, where $Hf(s) = \lim_{\varepsilon \to 0} (1/\pi) \int_{\varepsilon < |t-s| < \infty} f(t)/(s-t)\, dt$ a.e. $(s \in \mathbb{R})$.

Superreflexivity of Y is derived already from $L^\infty(Y) - L^1(Y)$-boundedness of the Y-valued Hilbert transform on the circle (=conjugate function operator, which is trivially equivalent to the analytic projection, too). In a similar vein, we have.

Proposition 3.4. Suppose $u^a \in N(Y)$ for all $u \in h\infty Y$. Then a *RNP* implies *RNP* for *Y*.

Proof. To derive *RNP* for Y, one has to show that every $u \in h^\infty(Y)$ has radial limits a.e. (Proposition 2.15.) Putting $v(z) := u(\bar{z})$, so that $v \in h^\infty(Y)$ as well, one easily obtains

$$u(z) = u^a(z) + {}^a u(z) - u(0) = u^a(z) + u^a(\bar{z}) - u(0).$$

By assumption, $u^a, v^a \in N(Y)$, and if Y has a *RNP* it follows that u^a, v^a have radial limits a.e. (2.17), whence the same holds for u.

Example 3.5.(L^1). The proposition says in other words that if $Y \in aRNP \setminus RNP$, then analytic projection cannot map $h^\infty(Y)$ into $N(Y)$. Moreover, the proof tells one how to produce examples: Take any $u \in h^\infty(Y)$ without a.e. existing boundary values, then necessarily $u^a \notin N(Y)$.

As a concrete example, consider $Y := L^1$ and $u: D \to L^1, u(re^{i\vartheta}) := P_r(\vartheta - \cdot)$. u is harmonic, e.g., by condition (ii) of 2.7, and $\|u(z)\|_Y = 1$ for all $z \in D$, thus $u \in h^\infty(Y)$.

Since the series expansion of u is $u(re^{i\vartheta}) = \sum_{n=-\infty}^{\infty} e_n r^{|n|} e^{in\vartheta}$, where $e_n \in L^1, e_n(\omega) = e^{in\omega}$, we have $u^a(z) = \sum_{n=0}^{\infty} e_n z^n = C_r(\vartheta - \cdot)(z = re^{i\vartheta})$,

so that

$$\|u^a(re^{i\vartheta})\|_Y = L^1 = \int_0^{2\pi} |C_r(\vartheta - \omega)| \frac{d\omega}{2\pi}$$

$$= \int_0^{2\pi} |C_r(t)| \frac{dt}{2\pi} =: \gamma_r.$$

γ_r does not depend on $\vartheta, \gamma_r \geq 1$, and as $r \to 1$,

$$\gamma_r = \int_0^{2\pi} \frac{1}{|1-re^{it}|} \frac{dt}{2\pi} \to \infty \quad (\sin ce \frac{1}{1-z} \notin H^1)$$

so that indeed $\int_0^{2\pi} \ln^+ \left\| u^a(re^{i\vartheta}) \right\|_Y (d\vartheta/2\pi)$

$= \ln \gamma_r \to \infty, i.e, u^a \notin N(L^1)$..

Keeping $Y = L^1$ fixed, we will show now that analytic projection is not a bounded operator $C(Y) \to N(Y)$ in the sense that $\sup_{f \in C(Y), \|f\|_\infty \leq 1} \left\| f^a \right\|_0 = \infty$

Take u as above and, for $R<1$, put $u_R(e^{it}) = u(R\, e^{it})$ then $u_R \in C(Y)$ with $\|u_R\|_\infty = 1$ for all R. On the other hand, as is easy to see, $(u_R)^a(z) = u^a(Rz)$..

Thus, as computed above,

$$\left\| (u_R)^a (re^{i\vartheta}) \right\|_Y = \left\| u^a(Rre^{i\vartheta}) \right\|_Y = \gamma_{Rr},$$

hence

$$\int_0^{2\pi} Ln^+ \left\| (u_R)^a (re^{i\vartheta}) \right\|_Y d\vartheta/2\pi = Ln\gamma_{Rr} \to Ln\gamma_R (r \to 1)$$

This means $\left\| (u_R)^a \right\|_0 = e^{\gamma R} \to \infty (R \to 1)$, as asserted.

As a corollary, analytic projection is not a bounded operator (and thus; by the closed graph theorem, not an operator at all) $C(L^1) \to H^p(L^1)$ for any $P > 0$, or, what amounts to the same, it is not $\|\cdot\|_\infty - \|\cdot\|_p$-bounded on the L^1-valued trigonometric polynomials. Note that even for $p = 1$ this does not follow directly from the result about superreflexivity quoted in Lemma 3.3, since the first part of its proof, proceeding along the lines of [31, 23.] works with step functions and thus outside $C(Y)$. For further examples of bad behaviour of the analytic projection see Examples 5.2,5.3&5.4.

4. The Dual Space of $\boldsymbol{H}^p(X)$

Let X be a complex Banach space and $1 < p < \infty, 1/p + 1/q = 1$ Recall the identifications

$$\boldsymbol{H}^p(X) = L_a^p(X) = \{f \in L^p(X): \hat{f}(n) = 0\ \forall n < 0\},$$
$$H^q(X') = L_a^q(X', X) = \{\mathrm{g} \in L^q(X', X): \hat{\mathrm{g}}(n) = 0\}.$$

We Define

$$\bar{H}^q(X') := \{\mathrm{g} \in L^q(X', X): \hat{\mathrm{g}}(n) = 0\ \forall n > 0\}.$$
$$H_0^q(X') := \{\mathrm{g} \in L^q(X', X): \hat{\mathrm{g}}(n) = 0\ \forall n \leq 0\}.$$

(Obviously, on the disc we have via Poisson integral

$$\bar{H}^q(X') := \{\mathrm{g}(\bar{z}): \mathrm{g}(z) \in H^q(X')\},$$
$$H_0^q(X') = \{\mathrm{g} \in H^q(X'): \mathrm{g}(0) = 0\}.)$$

The spaces $\bar{\boldsymbol{H}}^q(X')$ $\left(resp. \boldsymbol{H}_0^q(X')\right)$ are defined analogously, namely as $L^q(X') \cap \bar{H}^q(X')$ (resp. $L^q(X') \cap H_0^q(X')$).

Remark 4.1. By general Banach space theory, $\boldsymbol{H}^p(X)' = L^p(X)'/\boldsymbol{H}^p(X)^\perp$ Where $\boldsymbol{H}^p(X)^\perp$ is the annihilator of $\boldsymbol{H}^p(X)$ In $L^p(X)'$. In 2.1, $L^p(X)'$ was identified as $L^q(X', X)$, and $\boldsymbol{H}^p(X)^\perp \subset L^q(X', X)$ is easily recognized as $H_0^q(X')$, since analytic polynomials are dense in $\boldsymbol{H}^p(X)'$. We arrive at the description

$$\boldsymbol{H}^p(X)' = L^q(X', X)/H_0^q(X')$$

(canonically isometrically isomorphic), but of course one aims at a description of $\boldsymbol{H}^p(X)'$ as a space of functions, not equivalence classes.

Consider the canonical injective operators $\bar{H}^q(X') \to \boldsymbol{H}^p(X)', \mathrm{g} \mapsto \langle \cdot, \mathrm{g} \rangle = \int \langle \cdot, \mathrm{g} \rangle d\lambda$; $\bar{H}^q(X') \to L^q(X', X)/H_0^q(X')$.
which is the composition $\bar{H}^q(X') \hookrightarrow L^q(X', X) \twoheadrightarrow L^q(X', X)/H_0^q(X')$.

If X is a UMD space, then J is an isomorphism, since J^{-1} is then given by the antianalytic projection $\varphi \mapsto {}^a\varphi, L^q(X', X) \to \bar{H}^q(X')$ modulo its kernel $H_o^q(X')$. Vice versa, if J is an isomorphism , it is immediate to verify that $L^q(X', X) \twoheadrightarrow L^q(X', X)/H_0^q(X') \xrightarrow{\mathrm{J}-1} \bar{H}^q(X')$ is the antianalytic projection. We arrive at a theorem of Bukhvalov [8]: $\boldsymbol{H}^p(X)' \cong \bar{H}^q(X')$ canonically$\Leftrightarrow$ X admits analytic projection (p) $\Leftrightarrow$ X' admits [anti-] analytic projection (q) i.e., $X \in UMD$ (see 3.2, 3.3). The scalar multiplication in X' is to be understood as $(\lambda x') := \bar{\lambda} x'(x)$ $(\lambda \in \mathbb{C}, x \in X, x' \in X')$,[37]. This makes the dual pairing $\langle x, x' \rangle := x'(x)$ sesquilinear and allows one to replace $\bar{H}^q(X')$ by $H^q(X')$ in all of these consideration [8]. Alternatively, the latter effect could also be achieved by giving the dual pairing $\langle f, g \rangle$, defined as $\int \langle f(e^{it}), \mathrm{g}(e^{it}) \rangle d\lambda(t)$, here and in [8] $\left(f L^p(X), \mathrm{g} \in L^q(X', X)\right)$, the new meaning $\int \langle f(e^{it}), \mathrm{g}(e^{-it}) \rangle d\lambda(t)$, as in [10], similar to the case of Bergman spaces in [9].)

The problem arises to describe $\boldsymbol{H}^p(X)'$ for a general Banach space X as a space of functions–the more, since the ***UMD*** condition on X is extremely restrictive. The description (4.6) of $\boldsymbol{H}^p(X)'$ as $\bar{H}_*^q(X')$, a space of antianalytic X'- valued functions on the disc, is an attempt in this direction. Since the norm $\|\mathrm{g}\|_q^*$ of $\mathrm{g} \in \bar{H}_*^q\left(X'\right)$ depends rather explicitly on g's action as a functional or $\boldsymbol{H}^p(X)$, this answer is not really satisfactory. For instance, in the concrete case $X = c_0$ it does not yield an illuminating description of $\boldsymbol{H}^p(c_0)'$ but this might well be in the nature of things because of the bad behavior of l^1-valued analytic projection exhibited in Example 5.4. On the other hand, Bukhvalov's theorem mentioned above subordinates itself in a natural way as a special case (4.5), and some assertions about the position of $\bar{H}^q(X')$ in $\boldsymbol{H}^p(X)' = \bar{H}^q_*(X')$ can be made (4.10, 4.12).

In what follows, for a function f on D and $0 \le r < 1, f_r$ denotes function $f_r(z) = f(rz)$ on D and/or on T. If f is defined on T (and $P[f]$ makes sense), f_r means $P[f]_r = P_r * f$.

Lemma 4.2. Let $0 \le r < 1, f: D \to X, g: D \to X'$ harmonic with corresponding series expansions

$$f(re^{i\vartheta}) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\vartheta},$$

$$g(re^{i\vartheta}) = \sum_{n=-\infty}^{\infty} \hat{g}(n) r^{|n|} e^{in\vartheta}.$$

1^o $f \in \boldsymbol{h}^p(X) = L^p(X) \Rightarrow \langle f, g_r \rangle$

$$= \sum_{n=-\infty}^{\infty} \langle \hat{f}(-n), \hat{g}(n) \rangle r^{|n|}$$

2^o $g \in h^q(X') = L^q(X', X) \Rightarrow \langle f_r, g \rangle$

$$= \sum_{n=-\infty}^{\infty} \langle \hat{f}(n), \hat{g}(-n) \rangle r^{|n|}.$$

In particular, if $f \in L^p(X)$ and $g \in L^q(X', X)$ then $\langle f, g_r \rangle = \langle f_r, g \rangle$.

Proof. $f_r(re^{i\vartheta}) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\vartheta}$ with uniformly convergent series on T (r is fixed). Thus

$$\langle f_r, g \rangle = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \langle \hat{f}(n) r^{|n|} e^{in\vartheta}, g(e^{i\vartheta}) \rangle d\lambda(\vartheta)$$

$$= \sum_{n=-\infty}^{\infty} \langle \hat{f}(n), \hat{g}(-n) \rangle r^{|n|}.$$

The other equality is proved in the same way.

Corollary 4.3. Let $f \in L^p(X), g: D \to X'$ harmonic, $0 \le r, R < 1$.

1^o $\langle f, g_{Rr} \rangle = \langle f_R, g_r \rangle$

2^o $g \in h^q(X') = L^q(X', X) \Rightarrow \langle f, g \rangle = \lim_{r \to 1} \langle f, g_r \rangle$.

Proof. 1^o apply the lemma to g_r

2^o Follows from the lemma and $\langle f, g \rangle = \lim_{r \to 1} \langle f_r, g \rangle$; the latter because $f_r \to f$ in $L^p(X)$ (2.3).

Part 2 of this corollary says, in other words, that $g_r \to g$ as $r \to 1$ weak* in $h^q(X') = L^q(X', X) = L^p(X)'$, a fact which also follows directly from the general theory of Poisson integral representation [8].

Definition 4.4. Let $1 < q < \infty$, $1/p + 1/q = 1$. $\bar{H}^q_*(X') := \{g: D \to X' \ antianalytic: \|g\|^*_q := sup_{0 \le r < 1} \|g_r\|_{\boldsymbol{H}^p(X)'} < \infty\}$.

(Note that g_r is in $\bar{H}^q(X')$, thus in $\boldsymbol{H}^p(X)'$ after the discussion in 4.1)

Remarks 4.5. Let $g: D \to X'$ be antianalytic then:

1^o $\bar{H}^q_*(X')$ with $\|\cdot\|^*_q$ is a normed space (completeness will follow later).

2^o $\|g_r\|_{\boldsymbol{H}^p(X)'}$ increases to $\|g\|^*_q$ as $r \nearrow 1$.

3^o $\sup_{x \in X, \|x\| \le 1} \|\langle x, g \rangle\|_q \le A_q \|g\|^*_q \le A_q \|g\|_q$,

Where A_q is a constant independent of g (and X), and thus $\bar{H}^q(X') \subset \bar{H}^q_*(X')$

$\subset \{g: D \to X' \ antianalytic: sup_{\|x\| \le 1} \|\langle x, g \rangle\|_{\boldsymbol{q}} < \infty$, the first inclusion being continuous.

4^o $g \in \bar{H}^q(X') \Rightarrow \|g\|^*_q = \|g\|_{\boldsymbol{H}^p(X)'}$; in particular: $f \in \boldsymbol{H}^p(X), g \in \bar{H}^q(X') \Longrightarrow |\langle f, g \rangle| \le \|f\|_p \|g\|^*_q$.

5^o $\|g_r\|^*_q = \|g_r\|_{\boldsymbol{H}^p(X)'}$ increases to $\|g\|^*_q$ as $r \nearrow 1$ in particular, for $f \in \boldsymbol{H}^p(X), g \in \bar{H}^q_*(X'), r < 1$: $|\langle f, g_r \rangle| \le \|f\|_p \|g_r\|^*_q \le \|f\|_p \|g\|^*_q$.

Proof. 1^o If at all only "$\|g\|^*_q = 0 \Longrightarrow g = 0$" requires proof. $\|g\|^*_q = 0$ means $\langle \cdot, g_r \rangle = 0$ in $\boldsymbol{H}^p(X)'$ hence , $g_r = 0$ in $\bar{H}^q(X')$ (all r), the canonical map $\bar{H}^q(X') \to \boldsymbol{H}^p(X)'$ being injective. Hence $g(rz) = 0$ $\forall r \in [0,1)$ $\forall z \in D$, $i.e.$, $g = 0$.

2^o Take arbitrary r, $R \in [0,1)$. By Lemma 4.2, 1^o

$$\|g_{rR}\|_{H^p(X)'} = \sup_{\substack{f \in H^p(x) \\ \|f\|_p \le 1}} |\langle f_R, g_r \rangle|$$

$$\le \sup_{\substack{F \in H^p(x) \\ \|F\|_p \le 1}} |\langle f, g_r \rangle| = \|g_r\|_{H^p(X)'}.$$

3^o First inequality: fix $x \in X, \|X\| \le 1, r < 1.$ of course, $\langle x, g \rangle_r = \langle x, g_r \rangle \in \bar{H}^q \cong H^{p'}$ by scalar theory (or the discussion in 4.1). Hence (cf. [17, p. 113])

$$\|\langle x, g_r \rangle\|_q \le A_q \sup_{\substack{\varphi \in H^p \\ \|\varphi\|_p \le 1}} \left| \int \varphi \langle x, g_r \rangle d\lambda \right|$$

$$\le A_q \sup_{\substack{f \in H^p(X) \\ \|f\|_p \le 1}} \left| \int \langle f, g_r \rangle \, d\lambda \right|$$

$$= A_q \|g_r\|_{H^p(X)'}.$$

Now let $r \to 1$.

Second inequality:

$$\|g\|_q^* \le \sup_{r<1} \sup_{\substack{f\in L^p(X)\\ \|f\|_{p\le 1}}} |\langle f, g_r\rangle| = \sup_{r<1} \|g_r\|_q = \|g\|_q.$$

4^o By Lemma 4.2, if $g \in \bar{H}^q(X')$,

$$\|g\|_q^* \le \sup_{r<1} \sup_{\substack{f\in \boldsymbol{H}^p(X)\\ \|f\|_{p\le 1}}} |\langle f_r, g\rangle| \le \sup_{\substack{F\in \boldsymbol{H}^p(X)\\ \|F\|_{p\le 1}}} |\langle F, g\rangle| = \|g\|_{\boldsymbol{H}^p(X)'}.$$

On the other hand, by Lemma 4.2, 2^o

$$\|g\|_{\boldsymbol{H}^p(X)'} = \sup_{\substack{f\in \boldsymbol{H}^p(X)\\ \|f\|_{p\le 1}}} \left|\lim_{r\to 1}\langle f, g_r\rangle\right| \le \sup_{\substack{f\in \boldsymbol{H}^p(X)\\ \|f\|_{p\le 1}}} \sup_{r<1} |\langle f, g_r\rangle| = \|g\|_q^*.$$

5^o Apply 4^o to g_r; then 2^o.

Corollary 4.6. Let $\{g_i\}_{i=1}^{\infty}$, be a sequence where $g_i: D_j \to X_i'$ be antianalytic then,

i. $\bar{H}_*^{qj}(X_j)$ with $\|\cdot\|_{qj}^*$.

ii. $\sum \|(g_i)_r\|_{H^{pj}(X_j)'}$ increases to $\sum_{i=1}^{\infty}\|g_i\|_{qj}^*$ where $r \to 1$.

iii. $\sup_{\substack{x_i\in X_i\\ \|x_i\|\le 1}} \sum_{i=1}^{\infty}\|\langle x_j, g_i\rangle\|_{qi} \le A_{qi}\sum_{i=1}^{\infty}\|g_i\|_{qi}^* \le \sum_{i=1}^{\infty} A_{gi}$

such that A_{qj} is independent of g_i, X_j.

Hence $\bar{H}^{qj}(X_j') \subset \bar{H}_*^{qj}(X_j) \subset \{g_i: D_j \to X_j'\}$ an antianalytic and

$$\sup_{\|r\|\le 1} \sum_{i=1}^{\infty}\|\langle x_j, g_i\rangle\|_{qj} < \infty.$$

iv. $g_i \in \bar{H}^{qj}(X_j')$ implies that

$$\sum_{i=1}^{\infty}\|g_i\|_{q_i}^* = \sum_{i=1}^{\infty}\|g_i\|_{H^{p_j}(X_j)}.$$

For $f_j \in \bar{H}^{pj}(X_j), \sum_{i=1}^{\infty} g_i \in \bar{H}^{qj}(X_j')$, implies that

$$\sum_{i=1}^{\infty}\left|\left\langle f_j, g_i\right\rangle\right|_{q_j} \le \sum_{i=1}^{\infty}\left\|f_j\right\|_{p_j}\left\|g_i\right\|_{q_j}^*.$$

v. $\sum_{i=1}^{\infty}\|g_i\|_{qj}^* = \sum_{i=1}^{\infty}\|(g_i)_r\|_{H^{pj}(X_j)}$ increases to $\sum_{i=1}^{\infty}\|g_i\|_{qj}^*$ as $r \to \infty$.

For

$$f_j \in H^{pj}(X_j), \sum_{i=1}^{\infty} g_i \in \bar{H}_{qj}^*(X_j),$$

$$\sum_{i=1}^{\infty}\left|\left\langle f_j, g_i\right\rangle\right|_{q_j} \le \sum_{i=1}^{\infty}\left\|f_j\right\|_{p_j}\left\|(g_i)_r\right\|_{q_j}^* \le \sum_{i=1}^{\infty}\left\|f_j\right\|_{p_j}\left\|g_i\right\|_{q_j}^*.$$

Proof. i. If $\sum_{i=1}^{\infty}\|g_i\|_{q_j}^* = 0$ then $g_i = 0$. If $\sum_{i=1}^{\infty}\|g_i\|_{q_j}^* = 0$, it means that $\sum_{i=1}^{\infty}\left\langle \cdot, (g_i)_r\right\rangle = 0$ in $H^{pj}(X_j)'$ hence $(g_i)_r = 0$ in $\bar{H}^{qj}(X_j')$ and the map $\bar{H}^{qj}(X_j)' \to \bar{H}^{qj}(X_j')$ is injective.

That is $\sum_{i=1}^{\infty} g_i(r, z_j) = 0$ for every $0 \le r \le 1$ and $z_j \in D_j$ i.e. $g_i = 0$.

ii. For $0 \le r, R \le 1$, Corollary 4.3, shows that

$$\sum_{i=1}^{\infty}\|(g_i)_{rR}\|_{H^{pj}(X_j)'} = \sup_{\substack{f_j\in \boldsymbol{H}^{p_j}(X_j)\\ \|f_j\|_{p_j}\le 1}} \sum_{i=1}^{\infty}\left\|\langle (f_j)_R, (g_i)_r\rangle\right\|_{qj}.$$

Theorem 4.7. The map $\bar{H}_*^q(X') \to \boldsymbol{H}^p(X)', g \mapsto G$, where

$$Gf \lim_{r\to 1}\langle f, g_r\rangle = \lim_{r\to 1}\langle f_r, g_r\rangle \ \left(f \epsilon \boldsymbol{H}^p(X)\right)$$

is a (well-defined) isometric isomorphism.

Proof. First of all, for $f \in \boldsymbol{H}^p(X), g \in \bar{H}_*^q(X')$, by Remark 4.5, 5^o

$$|\langle f_r, g_r\rangle - \langle f, g_r\rangle| \le \|f_r - f\|_p \|g\|_q^* \to 0 \qquad (r\to 1)$$

since $f_r \to f$ in $L^p(X)$, as noted earlier. Thus we can dispose of the $\langle f_r, g_r\rangle$ version.

Now fix $g \in \bar{H}_*^q(X')$. For distinction, the functional $f \mapsto \langle f, g_r\rangle$ on $\boldsymbol{H}^p(X)$ (earlier identified with g_r) will be denoted by G_r $(0 \le r < 1)$. We have $\sup_{r<1}\|G_r\|_{\boldsymbol{H}^p(X)'} = \|g\|_q^* < \infty$.

If f is an analytic monomial $f(e^{i\vartheta}) = xe^{im\vartheta}$ $(x \in X, m \ge 0)$ then $\lim_{r\to 1} G_r f$ exists:

After Remark 4.5, 3^o $\langle x, g\rangle \in \bar{H}^q \subset L^q$ Whence

$$\langle f, g_r\rangle = fe^{im\vartheta}\langle x, g_r\rangle d\vartheta/2\pi = \langle x, g_r\rangle\hat{}(-m)$$

$$= r^m\langle x,g\rangle\hat{}(-m) \to \langle x,g\rangle\hat{}(-m) \text{ as } r\to 1.$$

Since analytic monomials form a total subset of $\boldsymbol{H}^p(X)$ (2.6), $Gf \lim_{r\to 1}\langle f, g_r\rangle$ exists for all $f \in \boldsymbol{H}^p(X), G \in \boldsymbol{H}^p(X)'$, and $\|G\| \le \|g\|_q^*$.

If $G = 0$, the calculation above yields $\langle x, g\rangle\hat{}(n) = 0 \quad \forall n \le 0, \forall x \in X$, hence

$\langle x, g\rangle = 0$ in $\bar{H}^q\ \forall x \in X$. This proves injectivity.

Surjectivity and other estimates: Let $G \in \boldsymbol{H}^p(X)'$ be given. Choose a Hahn-Banach extension $\tilde{G} \in L^p(X)', \|\tilde{G}\| = \|G\|$; by 2.1, $\tilde{G}$ is given by: $\tilde{g} \in L^q(X', X)$, $\|\tilde{g}\|_q = \|\tilde{G}\| = \|G\|$ put

$g := {}^a\tilde{g}: D \to X'$, the antianalytic projection of g. For

$f \in \boldsymbol{H}^p(X)$, by 4.2 and 4.3,

$$\langle f, g_r\rangle = \langle f, \tilde{g}_r\rangle \to \langle f, \tilde{g}\rangle = \tilde{G}f = Gf \qquad (r\to 1).$$

Thus g represents G and

$\|g\|_q^* = \sup_{r<1} sup_{f\in\boldsymbol{H}^p(X), \|f\|_p \le 1}|\langle f, g_r\rangle| \le \|G\|$,

which completes the proof.

In particular, $\bar{H}_*^q(X')$ is a Banach space. In terms of the canonical isometric isomorphism $L^q(X', X)/H_0^q(X') \longrightarrow \boldsymbol{H}^p(X)'$ (4.1), the proof yields.

Note 4.8. If $g \in \bar{H}^q(X')$, then g defines the functional $f \mapsto \langle f, g\rangle$ on $\boldsymbol{H}^p(X)$. On the other hand, by Remark 4.4, 3^o, $g \in \bar{H}_*^q(X')$as well and thus defines, after the theorem, the functional $f \mapsto \lim_{r\to 1}\langle f, g_r\rangle$. Fortunately, these two coincide, by Corollary 4.3, 2^o.

Corollary 4.9. $L^q(X',X)/H_0^q(X') \to \bar{H}_*^q(X'), [\varphi] \mapsto {}^a\varphi$

is an isometric isomorphism. In particular, $\bar{H}_*^q(X')$ consists exactly of the antianalytic projections of functions in $L^q(X', X) = h^q(X')$. (Here $[\cdot]$ denotes equivalence class mod $H_0^q(X')$.)

I want to show now that Bukhvalov's theorem already derived in Section 4.1 is contained in Theorem 4.7:

Corollary 4.10. (Bukhvalov) $\boldsymbol{H}^p(X)' \cong \bar{H}^q(X')$ under the canonical map (see 4.1) iff X admits analytic projection (p) $(i.e., X \in UMD, see\ 3.4)$.

Proof. In view of Theorem 4.7 (and note), $\boldsymbol{H}^p(X)' \cong \bar{H}^q(X')$ (canonically) iff $\bar{H}_*^q(X') = \bar{H}^q(X')$ as spaces of functions on the disc, with (then automatically (4.3, 3^o) equivalent norms $\|\cdot\|_q^* = \|\cdot\|_q$. Suppose this holds and let $\|\cdot\|_q \le C_q\|\cdot\|_q^*$. For any trigonometric polynomial $f \in L^p(X), f^a \in \boldsymbol{H}^p(X)$, whence

$$\|f^a\|_P = \sup_{\substack{g\in\bar{H}^q(X')\\ \|g\|_q^* \le 1}} |\langle f^a, g\rangle| \le \sup_{\substack{g\in\bar{H}^q(X')\\ \|g\|_q \le C_q}} |\langle f^a, g\rangle|$$

$$= \sup_{\substack{g\in\bar{H}^q(X')\\ \|g\|_q \le C_q}} |\langle f, g\rangle| \le C_q\|f\|_P$$

(last equality because g is of antianalytic type), so that X admits analytic projection (p). Conversely, if this latter condition is fulfilled with norm A_p, say, then for any $g: D \to X'$ antianalytic,

$$\|g\|_q = \sup_{r<1}\|g_r\|_q = \sup_{r<1} \sup_{\substack{f\in L^P(X)\\ \|f\|_P \le 1}} |\langle f, g_r\rangle|$$

$$= \sup_{r<1} \sup_{\substack{f\in L^P(X)\\ \|f\|_P \le 1}} |\langle f, {}^a g_r\rangle| \le \sup_{r<1} \sup_{\substack{F\in H^P(X)\\ |F|_P \le A_P}} |\langle F, g_r\rangle|$$

$$= A_P\|g\|_q^*,$$

so that $\bar{H}_*^q(X') = \bar{H}^q(X')$ (with equivalent norms).

Corollary 4.11. $H^{pj}(X_j)' \cong \bar{H}^{qj}(X_j')$ under the canonical map when X_j admits analytic projections (p_j).

Proof. Theorem 4.7 can show that $H^{pj}(X_j)' \cong \bar{H}^{qj}(X_j')$ if and only if $\bar{H}_*^{pj}(X_*^{qj}) \cong \bar{H}_*^{qj}(X_j')$ as spaces of functions on the disc with equivalent norm $\|\cdot\|_{qj}^* = \|\cdot\|_{qj}$. Now let $\|\cdot\|_{qj} \le \tilde{C}_{qj}\|\cdot\|_{qj}^*$ for any trigonometric polynomial sequence $f_j \in L^{pj}(X_j), f_j^a \in \boldsymbol{H}^{pj}(X_j)$, where

$$\|f_j^a\|_{p_j} = \sup_{\substack{g_1+\ldots+g_n\in\bar{H}_j^{qj}(X_j)\\ \|g_1+\ldots+g_n\|_{q_j}^* \le 1}} |\langle f_j^a, g_1+\ldots+g_n\rangle|$$

$$\leq \sup_{\substack{g_1+\ldots+g_n\in\overline{H}^{q_j}(X_j) \\ \|g_1+\ldots+g_n\|_{q_j}^* \leq \widetilde{C}_{q_j}}} \left|\left\langle f_j^a, g_1+\ldots+g_n\right\rangle\right|$$

$$= \sup_{\substack{g_1+\ldots+g_n\in\overline{H}^{q_j}(X_j') \\ \|g_1+\ldots+g_n\|_{q_j}^* \leq 1}} \left|\left\langle f_j, g_1+\ldots+g_n\right\rangle\right| \leq \widetilde{c}_{q_j}\left\|f_j\right\|_{p_j}.$$

Then X_j admits analytic projections (p_j).

Hence for any $g_1,\ldots,g_n : D \to X'$ antianalytics, then

$$\left\|g_1+\ldots+g_n\right\|_{q_j} = \sup_{r\leq 1}\left\|(g_1+\ldots+g_n)_r\right\|_{q_j}$$

$$= \sup_{r<1}\sup_{\substack{f_j\in L^{p_j}(X_j) \\ \|f_j\|_{p_j}\leq 1}} \left|\left\langle f_j, (g_1+\ldots+g_n)_r\right\rangle\right|$$

$$= \sup_{r<1}\sup_{\substack{f_j\in L^{p_j}(X_j) \\ \|f_j\|_{p_j}\leq 1}} \left|\left\langle f_j^q, (g_1+\ldots+g_n)_r\right\rangle\right|$$

$$\leq \sup_{r=1}\sup_{\substack{F\in H^{p_j}(X_j) \\ \|F_j\|_{p_j}\leq \widetilde{A}_{p_j}}} \left|\left\langle F_j, (g_1+\ldots+g_n)_r\right\rangle\right|$$

$$= \widetilde{A}_{p_j}\left\|g_1+\ldots+g_n\right\|_{q_j}^* = \widetilde{A}_{p_j}\left\|g_i\right\|_{q_j}^*$$

where $\widetilde{A}_{p_j}$ is a norm, so that $\overline{H}_*^{qj}(X_j') \cong \overline{H}^{qj}(X_j')$.

I continue with some assertions about the position of $\overline{H}^q(X')$ in $\overline{H}_*^q(X')$. As regards the weak* topology,

Proposition 4.12. 1^o If $g \in \overline{H}_*^q(X')$, then $\lim_{r\to 1} g_r = g$ in the weak* topology $\sigma\left(\overline{H}_*^q(X'), \boldsymbol{H}^p(X)\right)$, and $\|g_r\|_*^q \nearrow \|g\|_*^q$ as $r \nearrow 1$.

2^o Antianalytic polynomials are weak* sequentially dense in $\overline{H}_*^q(X')$. What is more, $B_{\overline{H}_*^q(X')} \cap$ {antianalytic polynomials} *is weak** sequentially dense in $B_{\overline{H}_*^q(X')}$.

Proof. 1^o Clear by Note 4.8 and Remark 4.5, 5^o.

2^o Note that weak* denseness alone of antianalytic polynomials in $\overline{H}_*^q(X')$ would follow already from the "abstract" criterion: Y a Banach (or locally convex) space, $V \subset Y'$ a vector subspace, then V is weak* dense in Y' iff $^{\perp}V := \{y\in Y : \langle y, y'\rangle = 0 \quad \forall y'\in V\} = 0$. Put here $y := \boldsymbol{H}^p(X), V :=$ {antianalytic polynomials}.

To prove (the second assertion of) 2^o take $g \in B_{\overline{H}_*^q(X')}$ and choose a sequence $r_n \nearrow 1$, then $g_{r_n} \in \overline{\boldsymbol{H}}^q(X'), g_{r_n} \to g$ weak* $(n \to \infty)$ and $\|g_{r_n}\|_q^* \leq \|g\|_q^* \leq 1$. Put $h_n := r_n g_{r_n}$, then also $h_n \in \overline{\boldsymbol{H}}^q(X'), h_n \to g$ weak* and $\|h_n\|_q^* < 1$, i.e., $h_n \in \dot{B}_{\overline{H}_*^q(X')} \cap \overline{\boldsymbol{H}}^q(X') \subset \overline{H}_*^q(X')$. This is a $\|\cdot\|_q$ open set in $\overline{\boldsymbol{H}}^q(X')$, because the inclusions $\overline{\boldsymbol{H}}^q(X') \subset \overline{H}^q(X') \subset \overline{H}_*^q(X')$ are continuous by 4.5, 3^o. Since antianaltyic polynomials are $\|\cdot\|_q$ dense in $\overline{\boldsymbol{H}}^q(X')$, we can choose one, say p_n, in $\dot{B}_{\overline{H}_*^q(X')}$ with $\|p_n - h_n\|_q \leq 1/n$. For $f \in \boldsymbol{H}^p(X)$ we have

$$|\langle f, P_n - h_n\rangle| \leq \|f\|_P \|p_n - h_n\|_q \leq (1/n)\|f\|_P \to 0,$$

so that $p_n \to g$ weak* in $\overline{H}_*^q(X')$ as well.

Corollary 4.13. $\boldsymbol{X'}$-valued antianalytic polynomials, equipped with $\|\cdot\|_q^*$, norm $\boldsymbol{H}^p(X)$, that is ,

$$\|f\|_p = sup\{|\langle f, g\rangle| : g : T \to X', \|g\|_q^* \leq 1\}$$

antianalytic polynomial, for all $f \in \boldsymbol{H}^p(X)$.

As regards the norm topology, we have

Theorem 4.14. 1^o If $\boldsymbol{X'}$ has RNP, then $g_r \to g$ for all $g \in \overline{H}_*^q(X')$..

2^o The following are equivalent:

(a) X' has *RNP*

(b) X' has a *RNP* and $\overline{H}^q(X')$ is dense in $\overline{H}_*^q(X')$

3^o The following are equivalent:

(a) $\overline{H}^q(X') = \overline{H}_*^q(X')$ $(i.e., X \in UMD)$

(b) $\overline{H}^q(X')$ is closed in $\overline{H}_*^q(X')$

(c) $\overline{\boldsymbol{H}}^q(X')$ is closed in $\overline{H}_*^q(X')$

Proof. 1^o By Corollary 4.7, the antianalytic projection $h \mapsto {}^a h$ is a bounded surjective operator $L^q(X', X) \to \overline{H}_*^q(X')$ [36] but if X'has RNP, then. $L^q(X', X) = \overline{H}_*^q(X')$

Now fix $g \in \overline{H}_*^q(X')$ take any $h \in L^q(X')$ with $g = {}^a h$, and use that $h_r \to h$ in $L^q(X')$. (2.3). It follows that $g_r = ({}^q h)_r = {}^a(h_r) \to {}^a h = g$.

2^o (a)⟹(b) Follows from 1^o. (b)⟹(a) by Corollary 4.7, we can identify $\overline{H}_*^q(X')$with $L^q(X', X)/\overline{H}_0^q(X')$. The density assumption then says that the canonical map $\overline{H}^q(X') \to L^q\ L^q(X', X)/H_0^q(X')$ has dense image. Since X' has a *RNP*, we have $H_0^q(X') = H_0^q(X') \subset L^q(X')$ (2.17) and it is clear that the map $i : L^q(X')/H_0^q(X') \to L^q(X', X)/H_0^q(X')$ is an isometry.

It follows that i has dense image as well and is thus surjectve . This means $L^q(X',X)$=$L^q(X')$+$H^q_o(X')$=$L^q(X')$ so that X' has RNP [36].

3^o (a) ⇒ (b) ⟹ (c)are trivial. (c) ⇒ (a) let $C_q$be a constant such that $\|\cdot\|_q \le C_q\|g\|^q_*$ over $\overline{\boldsymbol{H}}^q(X')$. For a trigonometric polynomial $f\in L^P(X), f^a \in H^P(X)$, whence by Corollary 4.12,

$$\left\|f^a\right\|_P = \sup_{\substack{g\in\overline{H}^q(X')\\ \|g\|^*_q\le 1}} \left|\langle f^a, g\rangle\right| \le C_q\|f\|_p$$

exactly as in the proof of Corollary 4.10, which also proves now (a)

Remarks 4.15. (a) Part 2^o shows that $\overline{H}^q(X')$ is in general not dense in $\overline{H}^q_*(X')$, e.g., certainly not if $X'\in a\ RNP\backslash RNP, e.g$ if $X=l^\infty, X'=L^\infty$.

(b) In other words, the canonical map $\overline{H}^q(X')\rightarrow L^q(X',X)/H^q_0(X')$ in general does not have dense image. In contrast to this, the analogous map $\overline{\boldsymbol{H}}^q(X') \to L^q(X')/\boldsymbol{H}^q_0(X')$ always has dense image--it contains all equivalence classes (mod $\boldsymbol{H}^q_0(X')$ of X'--valued trigonometric polynomials.

(c) In the proof of 2^o, we have had $H^q_0(X') = \boldsymbol{H}^q_0(X')$ and it was therefore trivial that $i: L^q(X')/\boldsymbol{H}^q_0(X') \to L^q(X',X)/H^q_0(X')$ is an isometry. I claim that this is always true, i.e., without the a RNP assumption on X':

Take $g\in L^q(X')$. since $g_r \to g$ in $L^q(X')$ after 2.3, one can write

$$\|g+\boldsymbol{H}^q_0(X')\|_{L^q(X',X)/H^q_0(X')}$$
$$= \inf_{h\in H^q_0(X')}\|g+h\|_{L^q(X',X)}$$
$$= \inf_{h\in H^q_0(X')}\sup_{r<1}\|g_r+h_r\|_q \ge \sup_{r<1}\inf_{H\in\boldsymbol{H}^q_0(X')}\|g_r+H\|_q$$
$$\ge \lim_{r\to 1}\left\|g_r+\boldsymbol{H}^q_0(X')\right\|_{L^q(X')/\boldsymbol{H}^q_0(X')}$$
$$= \|g+\boldsymbol{H}^q_0(X')\|_{L^q(X')/\boldsymbol{H}^q_0(X')}.$$

The reverse inequality being trivial, the claim is proved.

(d) Combining (b) and (c) yields: $\overline{\boldsymbol{H}}^q(X')$ is dense in $\overline{H}^q_*(X')$ iff i: $L^q(X')/\boldsymbol{H}^q_0(X') \to L^q(X',X)/H^q_0(X')$ is an isometric isomorphism.

Since in general, $\overline{H}^q_*(X') \supsetneq \overline{H}^q(X')$ and even $\overline{H}^q(X')$ functions on the disc posses "boundary values" $g^* \in L^q(X',X)$ on the circle only in a very weak sense, not much can be expected about boundary values of $\overline{H}^q_*(X')$ functions. Anyway, if $g\in\overline{H}^q_*(X')$, then $\langle x,g\rangle \in \overline{H}^q$ (4.5,3^o) with radial limit function $\langle x,g\rangle^* \in L^q$, for all $x\in X$. I will pursue the question if this collection of L^q functions $\langle x,g\rangle^*$ $(x\in X)$ give rise to single function $g^*: T\to X'$ with the property that for all $\langle x,g^*\rangle = \langle x,g\rangle^*$ a.e. (the exceptional set allowed to vary with x). (Of course, if $g\in\overline{H}^q(X')$, then its "boundary value") $g^*\in L^q(X',X)$ −the unique $g^*\in L^q(X',X)$ with $g = P[g^*]$ −does this job. But for a general $g\in\overline{H}^q_*(X')$, such a g^* −automatically scalarly measurable w.r.t. X −might exist without being in $L^q(X',X)$. The remote aim of this attempt would be, of course, to replace the action of the functional $g\in\overline{H}^q_*(X') = \boldsymbol{H}^{\boldsymbol{p}}(X)'$ as $\lim_{r\to 1}\int\langle f,g_r\rangle d\lambda\big(f\in\boldsymbol{H}^{\boldsymbol{p}}(X)\big)$ by a single integral $\int\langle f,g^*\rangle d\lambda$.

After Corollary 4.8, $\overline{H}^q_*(X') = \{{}^a h: h\in L^q(X',X)\}$. Fix $g = {}^a h \in \overline{H}^q_*(X') = \big(h\in L^q(X',X)\big)$. Then, for any function $g^*: T\to X'$, the condition $\forall x\in X: \langle x,g^*\rangle = \langle x,g\rangle^*$ a.e. is equivalent to saying $\forall x\in X: \langle x,g^*\rangle = \langle x, {}^a h\rangle^* = ({}^a\langle x,h\rangle)^* = {}^a\langle x,h\rangle$ a.e., where the last equality sign identifies the scalar $\overline{H}^q$ −function ${}^a\langle x,h\rangle$ with its boundary value.

In the following examples , it will be shown that, even for $h\in L^\infty(X',X)$, such a function $g^*: T\to X'$ need not exist. In these examples, $X = l^1, JT, c_0$. In the first one, h is even strongly measurable, that is, $h\in L^\infty(X') = L^\infty(l^\infty)$. Since $c_0' = l^1$ has **RNP**, thus $L^\infty(l', c_0) = L^\infty(l^1)$, this is naturally also the case in the last example. What makes this one more interesting is the fact that, due to the **RNP** of $X' = l^1$ and Theorem 4.14, $g_r \to g$ strongly in $\overline{H}^q_*(l^1)$ for all $g\in\overline{H}^q_*(l^1)$, and the boundary behavior of g can still be as bad as it can be.

It is of course equivalent to construct these examples with the analytic instead of the antiana1lytic projection.

5. Examples

Lemma 5.1. For $0\le s<t<2\pi$ and $z\in D$:

$$\operatorname{Im}\ \chi^a_{[s,t)}(z) = \frac{-1}{2\pi}\ln\left|\frac{e^{it}-z}{e^{is}-z}\right|.$$

Proof. This is an elementary calculation and, of course, well known.

I need some notation. The infinite dyadic tree is denoted by $tr := \{(m,j)\in\mathbb{N}^2_0: j\in 2^m\}$ [27]. For $(m,j)\in tr$ put $s_{mj} := j2^{-m}.2\pi, t_{mj} := (j+1)2^{-m}.2\pi$, so that $T_{mj} := (s_{mj}, t_{mj}) \subset [0,2\pi) = T$, is the jth dyadic interval of the mth generation. A number $\vartheta\in T$ is called dyadic if it is of the from $\vartheta = s_{mj}$ for some $(m,j)\in tr$. For $\vartheta\in T$ non–dyadic let $B_\vartheta := \{(m,j)\in tr: \vartheta\in T_{mj}\}$ be the "branch" of the tree associated with ϑ. Obviously,

$\bigcap_{(m,j)\in B_\vartheta} T_{mj} = \{\vartheta\}$. For $x = (x_\alpha)_{\alpha\in tr} \in \mathbb{C}^{tr}$ put $\mathrm{Im}\, x := (\mathrm{Im} x_\alpha)_{\alpha\in tr} \in \mathbb{R}^{tr}$.

Example 5.2. (c_0). There is $f \in L^\infty(c_0)$ such that.

(a) $lim_{r\to 1} f^a(re^{i\vartheta})$ exists for no $\vartheta \in T$ in $(l^\infty, \sigma(l^\infty, l^1))$. In particular, $f^a \notin N(c_0)$ because of Proposition 2.12;

(b) there exists no function $F: T \to l^\infty$ with the property $\forall x \in l^1: \langle x, F(\vartheta)\rangle = \langle x, f(\cdot)\rangle^a(\vartheta)$ a.e. (ϑ).

Proof . I realize c_0 as $\mathrm{c}_0(tr)$ and denote it again by c_0. Let $(\varepsilon_m)_{m\in\mathbb{N}_0}$ be a positive null sequence, which will be specified later.

Put $f: T \to \mathrm{c}_0, f(\vartheta) := \left(\varepsilon_m \chi_{[0,t_{mj})}(\vartheta)\right)_{(m,j)\in tr}$. By the Pettis measurability theorem [15], $f \in L^\infty(c_0)$. By Lemma 5.1,

$$\mathrm{Im} f^a(re^{i\vartheta}) = \left(\frac{-1}{2\pi}\varepsilon_m \ln\left|\frac{e^{it_{mj}} - re^{i\vartheta}}{1 - re^{i\vartheta}}\right|\right)_{(m,j)\in tr},$$

so that for ϑ non–dyadic,

$$\lim_{r\to 1} \mathrm{Im}\, f^a(re^{i\vartheta}) = \left(\frac{-1}{2\pi}\varepsilon_m \ln\left|\frac{e^{it_{mj}} - e^{i\vartheta}}{1 - e^{i\vartheta}}\right|\right)_{(m,j)\in tr} \in \mathbb{R}^{tr},$$

the limit taken coordinate-wise. (if ϑ is dyadic, the coordinate-wise radial limit does not exist). To prove (a) and (b), it suffices to choose (ε_m) in such a way that this last tuple does not blong to $l^\infty := l^\infty(tr)$, for all (non-dyadic) $\vartheta \in T$. ((ε_m) has to be independent of ϑ, of course.) Now fix $\vartheta \in T$ non-dyadic. Since always $\left(\varepsilon_m \ln|1 - e^{i\vartheta}|\right)_{(m,j)} \in l^\infty$, one only has to estimate $\left(\varepsilon_m \ln|e^{it_{mj}} - e^{i\vartheta}|\right)_{(m,j)\in tr}$, or the same expression only along $(m,j) \in B_\vartheta$. but for $(m,j) \in B_\vartheta$,

$$\varepsilon_m \ln|e^{it_{mj}} - e^{i\vartheta}| \le \varepsilon_m \ln|t_{mj} - \vartheta|$$
$$\le \varepsilon_m \ln(2^{-m}.2\pi) = -m\varepsilon_m \ln 2 + \varepsilon_m\, ln(2\pi) \to -\infty$$
$(m \to \infty)$,

e.g., for $\varepsilon_m := m^{-1/2}$,

The next example lives in the canonical predual B of the James tree space $JT = B'$ [27]. Since there is no lack of examples in more elementary Banach spaces.

Example 5.3. (B). There is $f \in L_B^\infty(JT', JT)$ such that

(a) $\lim_{r\to 1} \mathrm{Im}\, f^a(re^{i\vartheta})$ exists for no $\vartheta \in T$ in $(JT', \sigma(JT', JT))$ in particular, $f^a \notin N(B)$ because of Proposition 2.12 (note that $f^a = C[f]: D \to B$and that JT is separable [27]);

(b) there exists no function $F: T \to JT'$ with the property $\forall x \in JT: \langle x, F(\vartheta)\rangle = \langle x, f(\cdot)\rangle^a(\vartheta)$ a.e. (ϑ).

Example 5.4. (l^1)**.** There is $f \in (l^1)$ such that

(a) $lim_{r\to 1} f^a(re^{i\vartheta})$ exists for no $\vartheta \in T$ in $(l^1, \sigma(l^1, c_0))$. In particular, $f^a \notin N(l^1)$ because of Proposition 2.12;

(b) there exists no function $F: T \to l^1$ with the property $\forall x \in c_0: \langle x, F(\vartheta)\rangle = \langle x, f(\cdot)\rangle^a(\vartheta)$ a.e. (ϑ).

Proof . I realize l^1 as $l^1(tr)$ and denote it again by l^1. Let $(\varepsilon_m)_{m\in\mathbb{N}_0}$ be a positive summable sequence, which will be specified later.

Put $f: T \to l^1, f(\vartheta) := \left(\varepsilon_m \chi_{T_{mj}}(\vartheta)\right)_{(m,j)\in tr}$. It is clear that $f(\vartheta)$ is really in $l^1 = l^1(tr)$ for all ϑ and that $\|f(\vartheta)\|_1 = \sum_{m=0}^\infty \varepsilon_m$. Moreover, f is strongly measurable by Pettis theorem [15]; that is $f \in L^\infty(l^1)$ By Lemma 5.1,

$$\mathrm{Im} f^a(re^{i\vartheta}) = \left(\frac{-1}{2\pi}\varepsilon_m \ln\left|\frac{e^{it_{mj}} - re^{i\vartheta}}{1 - re^{i\vartheta}}\right|\right)_{(m,j)\in tr},$$

So that for ϑ non–dyadic ,

$$\lim_{r\to 1} \mathrm{Im}\, f^a(re^{i\vartheta}) = \left(\frac{-1}{2\pi}\varepsilon_m \ln\left|\frac{e^{it_{mj}} - e^{i\vartheta}}{e^{is_{mj}} - e^{i\vartheta}}\right|\right)_{(m,j)\in tr} \in \mathbb{R}^{tr},$$

the limit taken coordinate–wise. (If ϑ is dyadic, the coordinate-wise radial limit does not exist.) To prove (a) and (b), it suffices to choose (ε_m) in such a way that this last tuple does not belong to $l^1 = l^1(tr)$, for all (non-dyadic) $\vartheta \in T$. This will be achieved through the following.

Lemma 5.5. For ϑ non-dyadic, $m \ge 3$,

$$S_m(\vartheta) := \sum_{j}^{2^m - 1} \left|\ln\left|\frac{e^{it_{mj}} - e^{i\vartheta}}{e^{is_{mj}} - e^{i\vartheta}}\right|\right| \ge \ln\frac{2^m}{2\pi}$$

.

Accepting the lemma for a moment, we conclude as follows: fix $\vartheta \in T$ non-dyadic,

$$2\pi\left\|\lim_{r\to 1} \mathrm{Im}\ f^a(re^{i\vartheta})\right\|_1 \ge \sum_{m=3}^\infty \varepsilon_m S_m(\vartheta)$$
$$\ge \sum_{m=3}^\infty \varepsilon_m (m \ln 2 - \ln 2\pi)$$
$$= \ln(2)\sum_{m=3}^\infty m\ \varepsilon_m - \ln(2\pi)\sum_{m=3}^\infty \varepsilon_m = \infty ,$$

e.g, for $\varepsilon_m = m^{-2}$,

Proof. W.l.o.g., $\vartheta \in T_{m0}$ (if $\vartheta \in T_{mk}, 0 \le k < 2^m$, then putting $\tilde{\vartheta} := \vartheta - S_{mk} \in T_{m0}$ gives $S_m(\vartheta) = S_m(\tilde{\vartheta})$). Now

$$S_m(\vartheta) \ge \sum_{j=0}^{2^{m-1}-1} \left|\ln\left|\frac{e^{it_{mj}} - e^{i\vartheta}}{e^{is_{mj}} - e^{i\vartheta}}\right|\right|$$

$$\geq \sum_{j=0}^{2^{m-1}-1} \ln \left| \frac{e^{it_{mj}} - e^{i\vartheta}}{e^{is_{mj}} - e^{i\vartheta}} \right|$$

$$= \ln \left| e^{it_{m,2^{m-1}}} - e^{i\vartheta} \right| - \ln \left| e^{is_{m0}} - e^{i\vartheta} \right|$$

$$= \ln \left| -1 - e^{i\vartheta} \right| - \ln \left| 1 - e^{i\vartheta} \right|$$

$$\geq - \ln \left| 1 - e^{i\vartheta} \right|$$

$$\geq - \ln \vartheta$$

$$\geq - \ln \frac{2\pi}{2^m}.$$

since $\vartheta \in T_{m_0}$.

Note. Would we not have given away half of the terms in the first estimate, we could achieve the (irrelevant) improvement $S_m(\vartheta) \geq 2\ln(2^m / 2\pi)$.

Remark 5.6. Let $f \in L^{\infty}(L^1(tr))$ be the function just constructed. Since $C_r = \frac{1}{2}P_r + 1$ and f is coordinate-wise real, we have ${}^{a}f = \frac{1}{2}P[f] + \hat{f}(0) - i\ Imf^{a}$, and $Im\, f^{a}$ is the function just computed. Bearing in mind that, by Corollary 4.9, ${}^{a}f \in \overline{H}_{*}^{q}\left(l^1(tr)\right) = \boldsymbol{H}^p\left(c_0(tr)\right)'$ (all $q \in (1, \infty), 1/p + 1/q = 1$), there seems to be little hope for a simple description of $-\text{e.g.} -\boldsymbol{H}^p(c_0)'$.

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