Impact of Mathematical Programming Approach to Optimization using Fritz John Conditions

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Abstract: In mathematical programming it is customary to distinguish linear and convex programming. In nonlinear programming the objective function becomes nonlinear or one or more of the constraints inequalities have non-linear inequalities have non-linear relationship or both. Non-linear programming which has the problem of minimizing a convex objective function in the convex set of points is called convex programming where the constraints may taken to be non-linear.
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Introduction:

The Fritz John theorem is one of the most important results in mathematical programming. In linear programming the objective function and constraints are linear, in convex programming the objective functions admissible set are convex. Convex optimization problems are far more general than linear programming problems, but they share the desirable properties of LP problems: They can be solved quickly and reliably up to very large size up to hundreds of thousands of variables and constraints.

Nonlinear programming presents different perspective on mathematical programming problems in which the objective function and the constraint functions are not necessarily linear. There are many real world problems which have more than one conflicting objective functions. Such programming problems are called multiobjective programming problems. The mathematical discipline devoted to the theory and methods of finding the maximization and minimization of functions on sets defined by linear and nonlinear constraints. Mathematical Programming is a branch of optimization. It is used in various fields of man's activity where it is necessary to choose one course of action from several possible courses.

The general mathematical programming problem can be expressed as:

(P) Maximize (minimize) f(x)

Subject to $g_j(x)\ (\le,=,\ge)\ 0$, j=1,2,...,m

$x \in S$

Where f and g_j, j=1,2,…,m are real valued functions defined on $S \subseteq R^n$. The function f(x) is called constraints functions. In any given problem, the various members of (P) may have different equality/inequality signs but one and only one of the signs holds for each constraint.

When there are, besides inequality constraints, also equality constraints, the existing proofs are usually quite long and intricated. This is the case, for example, of the paper of Mangasarian and Fromovitz (1967), perhaps the first paper dealing with this topic, of the book of Bazaraa and Shetty (1967) and of Bazaraa, Sherali and Shetty (1993), of the paper of Still and Streng (1996), etc. An interesting paper of McShane (1973) uses the penalty approach and therefore it is useful in those courses on optimization, where also the computational aspects are treated[1].

In mathematical programming it is customary to distinguish linear and convex programming. In nonlinear programming the objective function becomes nonlinear or one or more of the constraints inequalities have non-linear inequalities have non-linear relationship or both. Non-linear programming which has the problem of minimizing a convex objective function in the convex set of points is called convex programming where the constraints may taken to be non-linear.

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HYPOTHESES FORMULATION

(a) The general mathematical programming problem can be formulated as:

Max (or min) f(x)

Subject to $g_j(x)\ (\le, =, \ge)\ 0,$ j=1,2,...,m

$x \in S$

Where f and g_j, j=1,2,...,m are real valued functions defined on $S \subseteq R^n$. The function f(x) is called the objective function and $g_j(x)$, j=1,2,...,m are called constraint functions.

(b) A general multiobjective programming problem having k (≥ 2) objectives is of the form:

(MP) min f(x) = ($f_1(x)$, $f_2(x)$, ... $f_k(x)$)

Subject to $g_j(x) \le 0,$ j=1,2,...,m

$x \in S$

Where f_i, i=1, 2,..., k and g_j, j=1,2,...,m are real valued functions defined on $S \subseteq R^n$ ..

(c)The mathematical representation of non-linear programming problem is as follows:

(P) Minimize f(x)

Subject to $g_j(x) \le 0,$ j=1,2,...,m

$x \in S$

where f and g_j ,j=1,2,…,m are real valued functions defined on $S \subseteq R^n$.

(d) In non-linear fractional programming we maximize (minimize) the ratio of two non-linear functions subject to linear or non-linear constraints. It is of the form;

$$\text{(FP)} \quad \text{maximize} \quad \frac{f(x)}{g(x)}$$

$$\text{subject to} \quad h_j(x) \le 0, \quad j = 1, 2, ..., m$$

$$x \in S$$

(FP) is said to be concave-convex fractional program, if f(x) is concave, g(x)is convex on the convex set S, if g is non-affine, then f is required to be non-negative. If f and g are differentiable, then concave convex fractional program has a pseudoconcave objective function.

Fritz-John [2] established necessary optimality conditions for the nonlinear programming problems without imposing any constraint qualification. **Mangasarian[3]** obtained necessary and sufficient conditions of optimality for nonlinear programming problems without assuming differentiability of the functions involved. He further derived Kuhn-Tucker's necessary optimality conditions under the weaker constraint qualification for pseudo-convex objective function and quasi-convex constraints.

In fractional programming problem if objective function is differentiable then cancave-convex fractional programming has a pseudoconcave objective function. Since the Kuhn-Tucker optimality conditions are often sufficient for a global optimal solution, therefore, cancave-convex fractional programming problem can be solved by various algorithms of convex programming. For **Frank-Wolfe's method [4]**, **Jagannathan [5]**, **Dinkelbach [6] and Geoffrion[7]** have shown that a fractional program can also be represented by a parametric program. **Dinkelbach [6]** proposed an iterative procedure that solves the equivalent parametric program. **Schaible [8]** modified Dinkelbach's algorithm and gave an algorithm similar to Dinkelbach's procedure and is based on a theorem by **Jagannathan [5]** concerning the relationship between fractional and parametric programming.

Proper efficiency of the solution of multi-objective programming problem is a strengthened solution concept. It eliminates unbounded trade-offs between the objectives. It was originally introduced by **Kuhn-Tucker[9]** and later followed by **Klinger[10],Geoffrion [11] and White[12]** for the usual multiobjective programming problem. The concept of efficiency was generalized to cone efficiency by **Yu[13].** Subsequently, proper efficiency was generalized by **Browien[14]**. Later the definition was strengthened by **Benson[15]** to assure equivalence to the Geoffrion definition even when the decision set is non-convex.

Optimality condition in Fritz John theorem

Optimality conditions are very important because they lead to the identification of optimal solutions. Fritz-John gave necessary optimality criteria for a non-linear programming problem without imposing any constraint qualification. They have proved that if x* is an optimal solution of (P), then there exists $r_0^* \in R, r^* \in R^m$, such that

$$r_0^* \nabla f(x^*) + r^{*T} \nabla g(x^*) = 0$$

$$r^{*T} \nabla g(x^*) = 0$$

$$(r_0^*, r^*) \ge 0$$

There is no guarantee that $r_0^* > 0$. In case when $r_0^* = 0$, the objective function f disappears from, the Fritz-John conditions and we have a degenerate case. In order to exclude such cases Kuhn-Tucker introduced restrictions on the constraints.

Duality in Fritz John theorem:

Duality plays a crucial role in mathematical programming. It is very useful to both theoretically and practically. A given problem of minimizations (maximization) subject to constraints, called primal problem, sometimes leads to another problem of maximization (minimization) subject to certain constraints known as a dual problem. The dual theorem states that for every minimization (maximization) problem called the primal problem, there is corresponding maximization (minimization) problem called the dual problem such that the minimum (maximum) value of former is equal to the maximum (minimum) value of the later.

Conclusion

Optimization is the act of obtaining the best result under given circumstances. In design, construction and maintenance of any decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables. Optimization can be defined as the process of finding the conditions that give the maximum or minimum value of a function. Optimization, in its broadest sense, can be applied to solve any engineering problem.

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