Derivation of an Analytic Expression for the Mass of an Individual Fish Larvae in an Uncapped Rate Stochastic Situation.

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ABSTRACT: There is need to vividly examine the impact of stochastic variations in diverse processes as may apply to a typical growth model. A capped rate stochastic process could be described as bounded by some limit and thorough delineation of the various factors affecting the growth of fish larvae is highly essential. The change in mass of fish larvae was considered due to physiological and metabolic pathways and other relevant factors to vividly examine the concept of stochastic process as applicable to the capped rate model, and delineate an analytically derived expression for the mass of an individual larvae based on relevant stochastic differential formulation in an uncapped rate stochastic process inundating the Ito lemma. The uncapped rate situation is only a removal of a maximum or imposed limit from a capped rate stochastic process.

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1.0 INTRODUCTION

Vast number of mathematical functions are known to take diverse forms and of various suitability to a number of daily life applications, ranging from biological, chemical to physical processes.

A number of processes defined by some relevant mathematical functions and equations can be modeled. A function can be defined such that its value do not exceed some capped or maximum imposed value. Interestingly, deterministic and stochastic processes are two well encountered cases, invariably, capped rate process on practical ground should incorporate some stochastic variation parameters.

The fish recruitment larvae have been considered in this case. An individual larva finds itself in a precarious situation, larvae exist in a highly stochastic and patchy environment and possess only limited locomotory and sensory ability. Larva are small relative to the spatial scales of prey heterogeneity and to the turbulent fluid flow at these spatial scales (Pitchford and Brindley, 2001). They also have only local knowledge of their immediate environment, limited by a visual perceptive distance of around one body length (Pitchford et al, 2003), and they are subject to massive mortality, with a newly hatched individual's probability of survival to metamorphosis being O(1%) or less (Chambers and Trippel, 1997) driven by typical mortality of 10% per day in the larva stage (Cushing and Horwood,1994). There are physiological limits on how fast an individual can grow, imposed by factors such as gut size and metabolism. A tiny minority of larvae even under favourable conditions survive to adulthood, thus 'average' larva is most definitely dead practically.

ΔM = min [prey encountered x conversion efficiency – metabolic cost maximum growth rate]Δt = min $[f_1(M)\, Z(M) - f_2(M);\, g(M)]\, \Delta t$ …………………… 1.0

where M represents the mass of an individual, Z(M) is the rate at which prey is encountered and digested (dependent on local prey concentration), $f_1(M)$ describes the efficiency of converting prey into mass, $f_2(M)$ represents metabolic costs and g(M) is the maximal rate at which individual can grow. Growth rate determines the duration of the period during which larvae are vulnerable to gape-limited predators (Fogarty et al, 1991).

Equation (1.0) above concisely describes a general model for an individual's change in mass, ΔM during time interval, Δt.

Stochastic and deterministic processes are frequently encountered in vast number of daily applications including physical, chemical and biological etc. Precisely, the growth of fish larvae is the subject of this discussion.

The concept of stochastic events or process cannot be taken with levity as it spans across most aspects of life and applications. Various forms of variations are encountered in one or more processes, indeed in sciences, finance to mention a few, we cannot but mention stochastic process. A capped rate process though has a maximum or limit imposed but practically, there are some fluctuations that worth being considered.

In real sense, fluctuations or variations exist, which take different forms, usually random appearing as noise. For instance, growth of an individual is influenced by internal and external factors. Internal factors are basically, physiological and metabolic pathways. The external factors would include factors like prey, predator, which are biological and physical factors like temperature, humidity, pH, salinity, light intensity etc. In a growth model, an individual is expected to grow continuously, realistically speaking, there is a limit or maximum growth limit. If the change in mass is considered with time, initially the growth rate depends on the physiological and metabolic pathways such as rate of conversion of food to mass and amount of food conversion utilized. However, there is a maximum or capped growth limit the individual can attain.

An ideal tool for this work is the Cushing-Horwood model of larval fish growth (Cushing & Horwood, 1994). According to the growth/mortality hypothesis (Cushing and Horwood, 1994; Rice et al, 1993), larvae which grow quickly through a ''mortality window'' have a survival advantage over those that do not (Campana, 1996).

Focusng on the fish larvae recruitment in this study, the capped rate stochastic process has been considered and the mass change considered in consonance with physiological factor, metabolic cost and time with other relevant factors. Even when a process is capped, there is expedient need to incorporate some stochasticity thereby unveiling a stochastic pattern.

2.0 Methodology

The Ito interpretation has been applied in delineating the capped rate process in fish larvae recruitment with consideration of some stochastic variables determining the change in mass of an individual fish larvae.

The model applied in this work is the Cushing-Horwood model (1994) with numerical treatment. The change in mass of the individual by this model is given by:

$$\Delta M = \min [(b(M) Z(M) - C(M) \times G(M)] \Delta t .$$

An analytic expression for the mass of an individual larvae has been derived.

3.0 Discussion

Many mathematical models which attempt to describe this process use continuous approximations specifically, an ordinary differential equation (ODE) is derived. There is an exigent need to incorporate stochastically in the ODE when uncertainty plays a significant role in the process, for example when prey are distributed patchly or the predator has a high mortality risk. However, the stochastic generalizations often rely on infinitesimally small time steps, not applicable to biological systems. Not including the ''unpredictable'' environment noise in fisheries models can lead (and has lead) to erroneous predictions of behavior of exploited stocks, and may have contributed to the deterioration of these stocks (Keyl and Wolff, 2008).

Deterministic models of recruitment can provide important insights into fish population dynamics in the face of exploitation (Fogarty, 1993). However, because the key natural phenomena are inherently stochastic, deterministic models can be argued to be inappropriate for qualifying recruitment. Rather, stochastic models should be constructed to arrive at recruitment probability (Pitchford and Brindley, 2001) and investigate recruitment variability (Forgarty, 1991, 1993).

There are physiological limits on how fast an individual can grow, imposed by factors such as gut size and metabolism, as much as 99.9% of larvae die before reaching metamorphosis (Campana, 1996), and rapid growth through the larva stage is thought to increase survival probabilities due to an increased ability to forage for prey and avoid predators (Cushing and Horwood, 1994).

Two naïve approaches to stochastic process comprises, the standard Weiner process and Poisson process. Where M(t) is defined as a diffusion process with constant drift;

$$dM(t) = \lambda dt + \sigma dW(t) \quad \ldots \ldots \quad 2.0.0$$

W(t) is a standard Weiner process (Grimmett and Stirzaker, 2001). With instantaneous mean zero,

Another approach; $dM(t) = dN_\lambda(t)$ …….. 2.0.1

Uses a Poisson noise process (Feller, 1950). As in equation 2.0, M(t) also has expectation λt in equation 2.0.1. These two stochastic process could be applied in principle to generalize equation (1.0) except where capped rate models are concerned, which could generate no leading results (James et al, 2005).

As a result of the stochastic process incorporated, a stochastic differential equation has a solution, which itself is a stochastic process. An earlier work related to Brownian motion was credited to Bachelier, 1900 in Theory of speculation and later followed up by Langevin Ito and Stratonovich.

3.1 Further Discussion

Considering the uncapped deterministic case

$$dM =((f_1(M) z(M) - f_2(M))dt \quad \ldots\ldots\ldots\ldots\ldots \quad 2.0.2$$

If z(M) is no longer deterministic, the infinitesimal rate z(M) would be replaced by z(M)dt + $\sigma_z(M)dW(t)$ where z(M) is an average value and $\sigma_z(M)$ measures the intensity of the stochastic perturbations by a Weiner process. Hence the SDE obtained is;

$$dM = f_1(M) z(M) dt + \sigma_z (M) dW(t) - f_2(M)dt \qquad 2.0.3$$

Equation 2.0.3 can be solved analytically or numerically depending on the forms of the various functions.

Stochastic differential equations are used extensively for instance in Physics as Langevin equation, probability theory and financial mathematics, numerical solutions etc.

A typical stochastic differential equation is of the form

$dX_t = \mu(X_t, t)\, dt + \sigma(X_t,t)dB_t$ ………….. 2.0.4

where B denotes a Weiner process (standard Brownian motion).

For functions fεN, the Ito integral is

$F[f](\omega)= \int_S^T f(t,\omega)dB_t(\omega)$ ………………. 2.0.6

In integral form, the equation is;δ

$X_{t+s} - X_t = \int_t^{t+s} \mu(X_\mu, u)du + \int_t^{t+s} \sigma(X_u, u)dBu$ …. 2.0.5

For functions fεN, the Ito integral is

$F[f](\omega)= \int_S^T f(t,\omega)dB_t(\omega)$ ………………. 2.0.6

where B_t is 1-dimensional Brownian motion.

The Stratonovich integral, is an alternative to the Ito integral. Unlike the Ito calculus, the chain rule of ordinary calculus applies to Stratonovich stochastic intergrals and the two can be converted viz the other for convenience as demanded.

A comparison of Ito and Stratonovich integral could be done.

The white noise equation;

$$\frac{dx}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \quad \text{………... 2.0.7}$$

has the solution X_t given as:

$$X_t = X_o + \int_o^t b(s,x)ds + \int_o^t \sigma(s, X_s)dB_s \quad 2.0.8$$

Adopting the Stratonovich equation as most appropriate in some cases, for t-continuously differentiable processes $B_t(n)$ such that:

$B^{(n)}(t,\omega) \to B(t,\omega)$ as $n \to \infty$.

Uniformly (in t) in bounded intervals. For each ω, let, $X_t^{(n)}(\omega)$ be the solution of the corresponding deterministic differential equation.

$$\frac{dx_t}{dt} = b(t, X_t) + \sigma(t, X_t)\frac{dB_t^{(n)}}{dt} \quad \text{….. 2.0.9}$$

$X_t^{(n)}(\omega)$ converges some function $X_t(\omega)$ and

$X_t^{(n)}(\omega) \to X_t(\omega)$ as $n \to \infty$, uniformly (in t) in bounded intervals.

The solution here, i.e. of the Stratonovich SDE

$$X_t = X_o + \int_o^t b(s, X_s)ds + \int_o^t \sigma(s, X) o dB_s \quad . 2.10$$

Coincides with the solution of the equation given by: 2.0.8 .

Thus, the solution of the modified Ito equation is;

$$X_t = X_o + \int_o^t b(s, X_s)ds + \frac{1}{2}\int_o^t \sigma(s, X)\sigma(s, X_s)ds + \int_o^t \sigma(s, X_s)dB_s \cdot 2.11$$

where σ denotes the derivative of σ (t, x) with respect to (Stratonovich, 1966).

Thus, the Stratonovich interpretation given by 2.11 should be more reasonably adopted compared to the Ito interpretation;

$$X_t = X_o + \int_o^t b(s, X_s)ds + \int_o^t \sigma(s, X)dB_s \quad \text{… 2.12}$$

which is the result of the white noise equation, 2.0.7.

Conversion between the Ito and Stratonovich integrals may be performed using the formula;

$$\int_o^T \sigma(X_t) o dW_t = \frac{1}{2}\int_o^T \sigma^1(X_t)dt + \int_o^T \sigma(X_t)dW_t \; .2.12(b)$$

Where X is some process, σ is a continuously differentiable function with derivative, σ^1 and the last is an Ito integral .

W(t) is used in deriving a stochastic differential equation. The change in an ass of the fish larva is considered in the current study. Each individual larva hatches with mass, M_o and grows according to the equation;

$\Delta M = \min((f_1(M)\, z(M) - f_2(M); G(M))\, \Delta t$ …… 2.13

Where $f_1(M)$ is the larva's efficiency, at converting food into biomass, $f_2(M)$ is the metabolic cost. Equation 2.13 is no longer an ODE and becomes a stochastic differential equations, SDE when the prey contact rate Z is defined as a random variable representing a heterogeneous food supply. With the defining function of the rate of change is not smooth in the minimum function of equation 2.13, Z follows a Gaussian distribution.

SDEs are solved analytically and computationally or numerically. The current study adopted the numerical method of solution to obtain results on the change in mass, ΔM of fish larva recruitment for two cases; capped rate and uncapped rate stochastic processes.

The uncapped deterministic case is;

$dM = (f_1(M)\, z(M) - f_2(M))dt$ ………. 2.14

If Z is no longer deterministic; the infinitesimal rate z(m) dt should be replaced by z(M) dt + σ(M)dW (t) where z(M) is an average value and $\sigma_z(M)$ measures the intensity of the stochastic perturbations; assumed to be given by a Weiner process. The SDE thus becomes;

$dM = f_1(M)\, z(M)\, dt + \sigma(M)dW(t) - f_2(M)\, dt$ 2.14(b).

This form of the SDE can be solves analytical or numerically depending on the form of the functions.

The time to maturity in this case has been extensively studied and some analytical solutions have been found, (Tarlin and Taylor, 1981, Grimmett and Stirzaker, 2001).

The deterministic capped – rate case is;

$dM = \min((f_1 (M) z(M) – f_2 (M), g(M))dt$ …. 2.15

quite different from the uncapped stochastic case.

Equation, 2.0.5 characteristics the behaviour of the continuous time stochastic process X_t as the sum of an ordinary Lebesque integral and an Ito integral. A plausible but very helpful interpretation of the stochastic differential equation is that in a small time interval of length d, the stochastic process X_t changes its value by an amount that is normally distributed with expectation $\mu(X_t, t)\delta$ and variance $\sigma (X_t,t)^2\delta$.The increments of the Weiner process are independent and normally distributed, thus this is independent of the past behaviour of the process.

The function μ is referred to as the drift coefficient, σ is called the diffusion coefficient and the stochastic process X_t is called the diffusion process.

3.2 Significance of the study

A typical economic scenario is dynamic often characterized by a number of variables. Some of the variables are predictable to a considerable extent while others are non-predictable often characterized by stochastic variations.

The concept of mathematical modeling kept on gaining ground in recent times. A typical model involves certain reasonable assumptions and consideration of variables that could be rarely static but usually dynamic over a time space. The concept of modelling has gained prominence in scientific use and now becomes of exigent need in most other recent daily applications, even stock exchange. Mathematical functions and statistical probability distribution functions are often vital tools in delineating the concept of capped-rate model in addition to other modelling applications. The idea is to impose a maximum value on a mathematical function that purports to explain the essential behaviours of the situation in peruse.

Delineating the uncapped rate stochastic process based and formulation of a model for the mass of an individual fish larva considering the Wiener formulation viz a stochastic differential equation becomes exigent in exploring the biological situation.

4.0 POPULATION GROWTH MODEL

The population growth model is

$$\frac{dN_t}{dt} = \alpha_t N_t;\ N_o \text{ given} \quad \ldots\ldots 2.21$$

Where $\alpha_t = r_t + \alpha.W_t$; $W_t \equiv$ white noise, α constant.

Assuming $r_t = r =$ constant. By the Ito interrelation, $\alpha(t,x) = \alpha x$ viz;

$$X_t = X_o + \int_o^t b(s, X_s)\, ds + \int_o^t \sigma(s, X_s) ds \quad 2.22$$

For the stochastic differential equation;

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t);\ b(t, x),\ x \in R,\ \sigma(t, x) \in R.$$

2.22b

where W_t is 1-dimensional "white noise"

$dN_t = rN_t dt + \alpha dB_t$

implies,

$$\int_o^t \frac{dN_s}{N_s} = rt + \alpha B_t\ (B_o = 0) \quad \ldots\ldots\ 2.23$$

using the Ito formula for the equation

$g(t,x) = \ln x;\ x > o$ …………………… 2.24

we have;

$$d(\ln N_t) = \frac{1}{N_t} . dN_t + {}^1\!/_2 \left(\frac{1}{N_t^2}\right)(dN_t)^2 \quad \ldots\ 2.25$$

$$= \frac{dN_t}{N_t} - \frac{1}{2}\alpha^2 dt$$

Hence;

$$\frac{dN_t}{N_t} = d(\ln N_t) + \frac{1}{2}\alpha^2 dt \quad \ldots\ldots 2.26$$

And thus; ln

$$\frac{N_t}{N_o} = \left(r - \frac{1}{2}\alpha^2\right)t + \alpha B_t \quad \ldots\ldots\ 2.27$$

or

$$N_t = N_o \exp\left(r - \frac{1}{2}\alpha^2\right)t + \alpha B_t \quad \ldots\ldots\ 2.28(b)$$

However, the stratonovich interpretation gives;

$$d\overline{N}_t = r\overline{N}_t\, dt + \alpha\, \overline{N}_t\, 0\, dB_t \quad \ldots\ldots 2.29$$

With the solution;

$$\overline{N}_t = N_o \exp(rt + \alpha B_t) \quad \ldots\ldots\ 2.30$$

The method of solution could be analytical or numerical depending on the forms of the various functions and solutions, where;

$\Delta M = \min [f_1 (M) z(M) – f_2 (M)g (M)] \Delta t$ …… 2.31

i.e. ΔM = min (prey countered x conversion efficiency – metabolic cost x growth rate) Δt

Removing the growth rate, i.e. the uncapped rate stochastic situation;

$dM = f_1 (M) z(M)dt – f_2 (M)dt$

If z(M) is stochastic then;

$dM = f_1 (M) z(M) dt + \sigma_z (M) dW(t) – f_2 (M) dt$ ..2.32

where z(M) in 2.32 is an average value.

$\sigma_z(M)$ measures the intensity of the stochastic perturbations driven by Weiner process. From 2.32;

$dM = f_1 (M) z(M)dt +\sigma_z(M)(M)dW(t) – f_2(M)dt$ 2.32b

To solve it analytically, let $W_o = 0$; $M(0) = M_o$
$dM = [f_1(M)\, z(M) - f_2\,(M)]dt + \sigma_z\,(M)dW(t)$ … 2.33

$\sigma_z(M) = \alpha M$, Oksendal.v. 5.1 using Ito interpretation.

$\therefore dM = [f_1\,(M)\, z(M) - f_2\,(M)dt + \alpha M dW(t)$ .. 2.34

For a smooth growth function i.e. an uncapped rate, it can be shown using a generalization of the central limit theorem that this formulation is equivalent to the SDE with $\sigma_z(M) = \sqrt{z(M)}$, provided the number of food items consumed is large (Feller, 1950, Whitt, 2002).

For instance, Binomial converges to Poisson as $N \to \infty$. For large N and $\lambda = z(M)$ as the mean arrival rate, as $\lambda \to \infty$, G(M) like Poisson: $\frac{e^{-\lambda x}}{\lambda}$ behaves suppressed.

$$\ln \frac{M}{M_o} = \left(r - \frac{1}{2}\alpha^2\right)t + \alpha W_t \quad \text{…………………2.35}$$

And thus; $M = M_o \exp\left[\left(r - \frac{1}{2}\alpha^2\right)t + \alpha W_t\right]$

viz integration of ordinary calculus and comparison with the Ito interpretation from the growth model in 2.28.

As an extension of the problem in vogue, the uncapped rate stochastic process has been added having mentioned the change in mass of the fish larval recruitment also for the capped rate stochastic process. The uncapped rate stochastic case differs by just a removal of the growth rate.

The time to maturity in this case has been studied extensively and some analytical solutions have been found (Karlin and Taylor, 1981; Grimmet and Stirzaker, 2001).

Having adapted the change in mass for fish larva recruitment viz the population growth model, we deduced an expression previously stated in comparison with the Ito interpretation for the individual mass, M_o to grow and becomes M viz;

$M = M_o \exp\left[r - \frac{1}{2}\alpha^2\right]t + \alpha W_t]$ viz integration of ordinary calculus.

4.1 Stochastic Differential Equation

Equation (1.0) above becomes a stochastic differential equation (SDE) and no longer an ODE when the prey encountered rate, Z is defined as a random variable representing a heterogeneous food supply.

Stochastic differential equations incorporate White noise, which can be seen as the derivative of Brownian motion or the Weiner process. It can also incorporate other types of random fluctuations such as jump process.

Firstly, the uncapped deterministic case is;
$dM = (f_1(M)\, Z(M) - f_2(M))dt$ …………… (2.18)

If Z(M) is no longer deterministic, the infinitesimal rate Z(M)dt should be replaced by Z(M)dt + $\sigma_z(M)$ dW(t), where Z(M) is an average value and $\sigma_z(M)$ measures the intensity of the stochastic perturbations (assumed to be driven by a Wiener process). The SDE is;
$dM = f_1(M)\, Z(M)dt + \sigma_z(M)dW(t) - f_2(M)dt$ . ..(2.19)

The SDE can be solved analytically or numerically depending on the forms of the various functions.

For the capped rate, we have;
$dM = \min\,(f_1(M)\, Z(M) - f_2(M);\, g(M)dt$ …… (2.20)

When metabolic cost, $f_2\,(M) = 0$, the time to maturity is simply the Nth arrival of the Poisson process, where N is calculated from M_{mat} and $f_1(M)$. In the case where $f_2\,(M) \neq 0$, the problem can be taken as that of a random walk hitting a moving barrier. From the central limit theorem, this is equivalent to the SDE with $\sigma_z(M) = \sqrt{Z(M)}$, provided the number of food items consumed is large (Feller, 1950; Whitt, 2002).

The change in mass is very sensitive to the time interval chosen when no limiting process is reached as $\Delta t \to 0$.

5.0 CONCLUSION

It is a worthwhile task exploring a growth model, precisely here fish recruitment larvae. The growth model plausibly, could be assumed to be capped. However, the attainment of capped rate stochastic situation might not be sufficient practically due to some stochastic or random fluctuations in real life processes, thus consideration of an uncapped rate situation becomes expedient. The stochastic variations in uncapped rate situation and the Wiener driven noise are practically inevitable.

For a typical growth model, change in mass is considered with respect to the individual mass and also vital for consideration are physical and physiological factors, especially in an uncapped rate stochastic situation. Specifically, metabolic cost, mean prey arrival rate, conversion efficiency of food, time interval were considered here. However, considering an Ito Interpretation, an analytical expression has been derived for determining the mass of an individual fish larvae in an uncapped rate stochastic process or situation.

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