**Bounded Operators with Imaginary Powers in Hilbert Space**

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**Abstract.** This paper deals with two aspects of the subject of the study. The first one consists of an operator of positive type in Hilbert space without bounded imaginary powers. The second one is concerned with the closedness of the sum of two closed operators in a Hilbert space. It shows the corresponding operators in with commuting resolvents and closable.

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1. **Introduction**

In a recent paper, Dore and Venni (G. Dore and A. Venni, 1987) have used imaginary powers of operators in connection with the problem of the closedness of the sum of two operators. Roughly speaking, if and are two commuting closed operators in a UMD-space, then their sum is closed provided that the following conditions holds:

The UMD–spaces are precisely the Banach spaces for which the vector valued. Hilbert transform is bounded in (J. Bourgain, 1983). In particular, the Hilbert spaces and –spaces, *1*, are UMD-spaces.

The growth condition (1.1) implies that the spectrum of  lies in a sector of "angle" .

In (G. Dore and A. Venni, 1987), the question was raised whether the converse is true. The Example below shows that this is not the case, even in a Hilbert space.

However, in a Hilbert space, the conditions for the closedness of the sum can be weakened, as shown again by Dore and Venni (G. Dore and A. Venni, 1987). Based on a characterization of the domain of fractional powers together with an earlier result of Da Prato and Grisvard (G. DA PRATO and P. GRISVARD, 1975), they proved the following result. If is a of bounded operators (without any assumption on ), then is closed provided that the sum of the “angles” and is less than .

In Example B, we give two operators and in a Hilbert space which satisfy the "angle condition" such that is not closed. This shows again that and are not of bounded operators. Moreover this implies that some extra condition is needed for the closedness of the sum .

In Section 2, we state the main results.

In Section 3, we interoduce the main tools for examples, in particular the notion of spectral family (E. Berkson and T. A. Gillespie, 1987).

In Section 4, we construct the example inspired by Example 5.10, p. 168, of Berkson and Gillespie (E. Berkson and T. A. Gillespie, 1987).

Finally, in section 5, we give Example , and corresponding operators in they resolvent commuting and closable. We are convinced that the method used in Sections 4 and 5 can lead to more examples.

1. **Preliminaries and main results**

Let be a complex Banach space, and let be a closed and densely defined operator with domain and range . As usual, we denote the resolvent set of by and its spectrum by .

The operator is called positive (G. Dore and A. Venni, 1987) if

1. there exists such that for every .

In particular, if , then is called .

For we define the as

The operator is said to be closable if it has an extension that is closed.

The operator is said to be of type (H. TANABE, 1979), if there exist and such that;

1. ;
2. for every there exists with , such that for any .

We recall that if the operator is positive, then there exist and such that is of type (H. TRIEBEL, 1978).

We also recall that if is –accretive, then is of type (H. TANABE, 1979). Moreover if *A* is of type for some and , then generates an analytic semigroup on the space .

If is abounded positive operator with , then the fractional powers of denoted by  with are usually defined by the Dunford integral

Where the *contour*  does not meet and contains the spectrum of . Then for  *,*  is a bounded operator satisfying the group property

The function is also holomorphic. Moreover, one has the other representations of (J. PRÜB and H. SOHR, 1990),

or equivalently

If the positive operator satisfies only and dense in , then for every which is dense in , the function , defined by (2.1) or (2.2) is holomorphic and satisfies the group property

For we say that is bounded if the operator defined by (2.1) or (2.2) is bounded on . Then it can be uniquely extended to , as a bounded operator.

Following and Sohr (J. PRÜB and H. SOHR, 1990), the operator is said to belong to the class for some if :

1. is positive;
2. ;
3. and there exists such that

In the case where is positive, implies the density of in if is a reflexive Banch space (a Hilbert space, for example).

It is proven in (J. PRÜB and H. SOHR, 1990), that if then is of type  for some . In Example , we show in particular that the converse is not true even if the space is a Hilbert space.

**Example A.** There exists an operator *A* in a Hilbert space which is of type for some and for all and such that the imaginary powers are not bounded for all .

**Remark.** It is known (J. PRÜB and H. SOHR, 1990) that if an operator *A* in Hilbert space is of type *(, 1)* for some (it is *m*-accretive), then .

Let *A* and *B* be two positive operators in a Banach space . The operators *A* and *B* are called if and commute for some and (equivalently for all and ) .

Building upon results of Dore and Venni (G. Dore and A. Venni, 1987), and Sohr (J. PRÜB and H. SOHR, 1990) have proven that if , are resolvent commuting and if is a UMD-space, then where  *=* max *()*.

Da Prato and Grisvard ( G. DA PRATO and P. GRISVARD, 1975) have proved that if are of typeresolvent commuting ( hence closable ) then the closure of is of type with = max*()* for some .

Therefore a natural question is to know whether the sum of two operators and satisfying the assumptions of Da Prato and Grisvard in a UMD-space is closed. In the Hilbert space, Da Prato and Grisvard ( G. DA PRATO and P. GRISVARD, 1975) gave a sufficient condition for this to be the case, namely if the interpolation spaces and are equal for some .Since is closed if and only if is closed, we may assume without loss of generality that and . Under these assumptions Dore and Venni (G. Dore and A. Venni, 1987. p. 194), have shown that if the imaginary powers is are uniformly bounded for , then is closed .

**Example B.** There exists two resolvent commuting operators and in aHilbert space which are of type for some and for every such that is not closed.

**Remarks.** (i) It follows from Da Prato and Grisvard (G. DA PRATO and P. GRISVARD, 1975) that and for every .

(ii) It follows from Dore and Venni (G. Dore and A. Venni, 1987) that both and are not uniformly bounded on .

1. **Tools**

We recall the notion of spectral family of projections in a Hilbert space (E. Berkson and T. A. Gillespie, 1987).

Definition. Aspectral family of projections in is a uniformaly bounded projection–valued function ( the algebra of bounded linear operators in ) such that:

1. is right–continuous in the strong operator topology,
2. has a strong left–hand limit at each ,
3. in the strong operator topology as .

If there is a compact interval such that  for and for , then we say that is concentrated on . Following (E. Berkson and T. A. Gillespie, 1987), (H. R. DOWSON, 1987), if is a spectral family concentrated on , each complex–valued function defines abounded operator in stands for bounded variation) :

by means of convergence of Riemann-Stieltjes sums. Moreover the norm of can be estimated by

Where

If is concentrated on and *,* then exists. This limit defines abounded operator in satisfying.

Where is defined by (3.3) and which exists since .

If and

then

If moreover , then

If , for every and belongs to , then and

For the construction of a spectral family in which is not spectral measure, we shall use, as in (E. Berkson and T. A. Gillespie, 1987), a conditional basis which can be found in Singer ( I. SINGER, 1970). For the sake of completeness, we give it here explicitly.

The sequences  and  in defined by

Where  is the canonical basis of and , (e.g., one can take are biorthogonal conditional bases of . Defining by

Where is the scalar product, then each aprojection with for satisfying

Moreover

1. **Example A**

we construct an example of appositive operator in a Hilbert space such that imaginary powers are not bounded for *,* although  is of type for some and for every .

In order to do that, we construct the operator on a Hilbert product.

Let be a family of complex Hilbert spaces. Let be the Hilbert product.

The family of bounded operators on , defines the following closed densely defined operator on :

Moreover is bounded if and only if and if this is the case

.

We say that family of positive operators satisfies if:

1. for every , there is independent of , such that for every and every

We have

**Lemma 4.1.** Let be a family of bounded positive operators on (P) then there exists such that the operator defined by (4.1), is of type  for every .

Moreover if , then for every and , we have and .

**Proof .** (i) Let and let. Since satisfies Property (P), and there exists such that

Since we have Moreover since , we have and This implies that is of type with , for every

(ii) Assume  then for every . Let Then clearly, . Since for some , we have , hence Therefore and are well–defined by (2.1), for . Since *,* we obtain This completes the proof of Lemma 4.1.

Next we construct a family of bounded positive operators  in , such that and satisfying Properly . Notice that the imaginary powers , are then bounded. We give a necessary condition for  to be finite for some .

**Lemma 4.2.** Let be a (Schauder) basis on with corresponding projections .

Let be the spectral family concentrated on defined by

.

Then for every  and every

is well–defined

and

1. The family of operators  satisfies Property and
2. For every , the imaginary power is bounded and *,* . Moreover
3. If for some  then the basis is unconditional.
4. If the basis is unconditional then for all .

**Proof .** (i) For every  the function is continuous, bounded, increasing, hence of bounded variation on . Therefore is well-defined and bounded on as well as . Moreover .

Let and. Then the function is continuous, bounded, and of bounded variation on [0,1]. Indeed then where

Moreover

with

Let . We observe that and increases on .

Therefore which implies that the family satisfies Property .

(ii) Let then  and

Hence defines a bounded operator for every and . For (finite sequences in , we have

for some depending on .

By using the Dunford integral for the imaginary power we obtain

Since both and are bounded on and is dense in we have . We also have .

1. If for some then and without loss of generality, we may assume . We also have . By using a result of Nagy (B. SZ-NAGY, 1947) , there exists an equivalent Hilbertian norm on such that for every . (Take, e.g.,where Lim is a Banach limit in ℕ.) Then is unitary in and are eigenvectors corresponding to the eigenvalues

Then for we have . Therefore is an orthogonal system in , hence is an unconditional basis in and also in .

1. Suppose the basis is unconditional. By using a characterization of unconditional bases, there exists a constant such that for every and every finite scalar sequence

For (the linear dense subspace spanned by , we have

the sum is finite. Hence

. Then .

After these preparations, we can easily construct the operator .

Construction of . Let and let be a conditional basis of , for example, the basis defined in (3.5). Define like in Lemma 4.2, then for every . Then define the operator like in Lemma 4.1. The operator is of type for some , and for every  Moreover for  cannot be bounded, otherwise would be finite. There for the operator satisfies all the required properties.

1. **Example B**

In this section, we construct an example of two resolvent commuting, closed operators and , in a Hilbert space such that and are of type for some and every with not closed.0

Let be a (Schauder) basis in , and be the associated projections.

We shall denote by the linear dense subspace spanned by .

Let be the spectral family defined by

where denotes the greatest integer .

We define *.*

**Lemma 5.1.** Let , and be as above. Let be a continuous and increasing function. For any let

Then, for every there exists such that for every is a bijection in and

Moreover is closable and its closure is of type  for some  for every and satisfies .

Proof of Lemma 5.1 . (i) Proof of (5.2). For every , we define

. We get The spectral representation of is given by

By using (3.4), we have

for every  which may be infinite.

with .

Then we get (5.2).

(ii) Closure of . It is known, see, e.g., ( G. DA PRATO and P. GRISVARD, 1975), that (5.2) implies that  closable and that its closure satisfies the same inequality. For the sake of completeness, we prove that closable.

Let be such that and for some . We have to prove . Let then for , we have and by taking the limit. Hence and  by letting for every . Since is dense in .

(iii) Type of . From (5.2), we get for every and , which implies that is injective and that is closed, hence Therefore and holds for every .

(iv) . Let . is the inverse of by using (3.4), we get

Then  is bounded and densily defined. This implies that the closure of  is the inverse of .

Next, we consider properties of two operators and of the form given by Lemma 5.1.

**Lemma 5.2.** Let and be two continuous, increasing functions from into [1,. Let and be the corresponding operators in defined by

Let and be their closure in .

Then, we have

1. and are resolvent commuting;
2. is closable and

**Proof .** (i) We have

. Since is a bijection on , it follows that and commute.

(ii) As is well known it suffices to prove . But this is a consequence of the commutativity of together with their boundedness.

(iii) First we prove that is closable. Let be such that and with Then

Hence .

Since the closure of is contained in the closure of , we only have to prove or . Let Then there are tow sequences .

Set . We have

by using part (i). Since  is bounded by Lemma 5.1, we obtain that the sequence converges to some Moreover , then since is closable by Lemma 5.1. Rewriting (5.3), we get

which implies by passing to the limit

**Corollary 5.3.** Let be two increasing continuous sequences of functions from into . Let and be the corresponding operators in defined by



and

Let  *,*  be their closure in then we have



on .

and are resolvent commuting .

is closable and .

**Proof .** Lemma 5.2 implies that



since *x* is total we have . It follows that, , since is a bijection, then , implies that .









hence .

Follows directly from Lemma 5.2.

Now we give a Lemma which characterizes the closedness of .

**Lemma 5.4**. Let the operators and be defined as in Lemma 5.2. Then is not closed if and only if there exists a sequence in such that

**Proof .** (i) Let . We define two norms on :

Clearly for . and are closed, is complete with respect to the norm . Moreover is complete with respect to if and only if  is closed. By using the open mapping theorem (for one implication), one has is closed if and only if there exists such that

(ii) Let be such that with

Then (5.6) cannot hold. Indeed, we have

and

which is unbounded.

Hence is not closed .

(iii) Assume . By triangular inequality, there is such that

Then if we have

*.*

Then the norms and are equivalent on . Observe that . which is dense in with respect to the norm Notice that

Hence is dense on with respect to . For there exists such that and , by using the continuity of on . It follows that the norm and are equivalent on .

Construction of the Example B. It is enough to choose  and as in Lemma 5.1 and 5.2 such that condition (5.5) of Lemma 5.3 is satisfied, i.e., to find tow functions and  as in Lemma 5.1 such that

We show that this is possible .

First we choose for the conditional basis of example (3.5) which satisfies

.

If we impose the following conditions on and ,

then

which satisfies (5.7).

Finally, we give one possible choice of functions  and satisfying the hypothesis of Lemma 5.1 and condition (5.8).

Set

We contract and by induction :

and

Suppose we know the functions between  *,*  then we define for

and for

Then, are continuous on , nondecreasing, not less than one with

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