Bounded Operators with Imaginary Powers in Hilbert Space

Ahmed Alnaji Abasher

Department of Mathematics, College of Science and Arts, Shaqra University, Saudi Arabia
ahmednb99@hotmail.com

Abstract. This paper deals with two aspects of the subject of the study. The first one consists of an operator of positive type in Hilbert space without bounded imaginary powers. The second one is concerned with the closedness of the sum of two closed operators in a Hilbert space. It shows the corresponding operators in H_0 with commuting resolvents and closable.
[Ahmed Alnaji Abasher. **Bounded Operators with Imaginary Powers in Hilbert Space.** *N Y Sci J* 2014;7(4):24-31]. (ISSN: 1554-0200). http://www.sciencepub.net/newyork. 5

Keywords: Hilbert space; imaginary power; H_0 ; commuting

1. Introduction

In a recent paper, Dore and Venni (G. Dore and A. Venni, 1987) have used imaginary powers of operators in connection with the problem of the closedness of the sum of two operators. Roughly speaking, if A and B are two commuting closed operators in a UMD-space, then their sum is closed provided that the following conditions holds:

$$\|A^{is}\| \leq Me^{\omega A|s|} \text{ and } \|B^{is}\| \leq Me^{\omega B|s|}, s \in \mathbb{R} \quad (1.1)$$
$$\text{with } \omega_A + \omega_B < \pi.$$

The UMD–spaces are precisely the Banach spaces X for which the vector valued. Hilbert transform is bounded in $L^2(\mathbb{R}; X)$ (J. Bourgain, 1983). In particular, the Hilbert spaces and L^P – spaces, $1 < p < \infty$, are UMD-spaces.

The growth condition (1.1) implies that the spectrum of A $(resp.\ B)$ lies in a sector of "angle" $\omega_A (resp.\ \omega_B)$.

In (G. Dore and A. Venni, 1987), the question was raised whether the converse is true. The Example A below shows that this is not the case, even in a Hilbert space.

However, in a Hilbert space, the conditions for the closedness of the sum can be weakened, as shown again by Dore and Venni (G. Dore and A. Venni, 1987). Based on a characterization of the domain of fractional powers together with an earlier result of Da Prato and Grisvard (G. DA PRATO and P. GRISVARD, 1975), they proved the following result. If A^{is} is a c_0-group of bounded operators (without any assumption on B^{is}), then $A + B$ is closed provided that the sum of the “angles” w_A and w_B is less than π.

In Example B, we give two operators A and B in a Hilbert space which satisfy the "angle condition" such that $A + B$ is not closed. This shows again that A^{is} and B^{is} are not c_0-groups of bounded operators. Moreover this implies that some extra condition is needed for the closedness of the sum .

In Section 2, we state the main results.

In Section 3, we interoduce the main tools for examples, in particular the notion of spectral family (E. Berkson and T. A. Gillespie, 1987).

In Section 4, we construct the example A inspired by Example 5.10, p. 168, of Berkson and Gillespie (E. Berkson and T. A. Gillespie, 1987).

Finally, in section 5, we give Example B, and corresponding operators in H_0 they resolvent commuting and closable. We are convinced that the method used in Sections 4 and 5 can lead to more examples.

2. Preliminaries and main results

Let $(X, \|.\|)$ be a complex Banach space, and let $A: D(A) \subset X \to X$ be a closed and densely defined operator with domain $D(A)$ and range $R(A)$. As usual, we denote the resolvent set of A by $\rho(A)$ and its spectrum by $\sigma(A)$.

The operator A is called positive (G. Dore and A. Venni, 1987) if

(i) $(-\infty, 0) \subset \rho(A)$,

(ii) there exists $M \geq 1$such that $\|(I + tA)^{-1}\| \leq M$, for every $t > 0$.

In particular, if $M = 1$, then A is called m-accretive.

For $\theta \in [0, \pi)$, we define the sector Σ_θ as

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\}; |arg\ z| \leq \theta\}.$$

The operator A is said to be closable if it has an extension that is closed.

The operator A is said to be of type (ω, M) (H. TANABE, 1979), if there exist $0 < \omega < \pi$ and $M \geq 1$ such that;

(i) $\sigma(A) \subset \Sigma_\omega \cup \{0\}$;

(ii) for every $\theta \in [0, \pi - \omega)$, there exists $M(\theta) \geq 1$ with $M(0) = M$, such that $\|(1 + zA)^{-1}\| \leq M(\theta)$ for any $z \in \Sigma_\theta$.

We recall that if the operator A is positive, then there exist $\theta \in (0,\pi)$ and $M \geq 1$ such that A is of type (θ, M) (H. TRIEBEL, 1978).

We also recall that if A is M–accretive, then A is of type $(\pi/2, 1)$ (H. TANABE, 1979). Moreover if A is of type (ω, M) for some $\omega \in (0,\pi/2)$ and $M \geq 1$, then $-A$ generates an analytic semigroup on the space X.

If A is abounded positive operator with $0 \in \rho(A)$, then the fractional powers of A denoted by A^z with $z \in \mathbb{C}$ are usually defined by the Dunford integral

$$A^z = \frac{1}{2i\pi}\int_\Gamma \lambda^z(\lambda - A)^{-1}d\lambda\,,$$

Where the *contour* Γ does not meet $(-\infty, 0]$ and contains the spectrum of A. Then for $z \in \mathbb{C}$, A^z is a bounded operator satisfying the group property

$$A^{z_1+z_2} = A^{z_1}A^{z_2}\,,\ z_1, z_2 \in C\,,$$

with $A^0 = I$ and $A^1 = A$.

The function $z \mapsto A^z$ is also holomorphic. Moreover, one has the other representations of A^z(J. PRÜB and H. SOHR, 1990),

$$A^z x = \frac{sin\pi z}{\pi}\Big\{z^{-1}x - (1+z)^{-1}A^{-1}x + \int_0^1 t^{z+1}(t+A)^{-1}A^{-1}x dt + \int_1^\infty t^{z-1}(t+A)^{-1}Axdt\Big\} \qquad (2.1)$$

$$\forall\, |Re\, z| < 1\,, z \neq 0,$$

$$A^0 x = x$$

or equivalently

$$A^z x = \frac{sin\pi z}{\pi}\Big\{z^{-1}x - (1+z)^{-1}A^{-1}x + +(1-z)^{-1}Ax + \int_0^1 t^z(1+t^{-1}A)^{-1}A^{-1}x\, dt - \int_0^1 t^{-z}(1+t^{-1}A^{-1})^{-1}Axdt\Big\} \qquad (2.2)$$

$$\forall\ |Re\, z| < 1, \quad z \neq 0\,,$$

$$A^0 x = x.$$

If the positive operator A satisfies only $N(A) = \{0\}$ and $R(A)$ dense in X, then for every $x \in D(A) \cap R(A)$, which is dense in X, the function $z \mapsto A^z x$, defined by (2.1) or (2.2) is holomorphic and satisfies the group property $A^{z_1+z_2}x = A^{z_1}A^{z_2}x\ \ for\ every\ \ x \in D(A^2) \cap R(A^2)$ and $[|Re\, z_1|, |Re\, z_2|,$ $|Re(z_1 + z_2)| < 1$ (J. PRÜB and H. SOHR, 1990).

For $s \in \mathbb{R} \setminus \{0\}$ we say that A^{is} is bounded if the operator A^{is} defined by (2.1) or (2.2) is bounded on $D(A) \cap R(A)$. Then it can be uniquely extended to X, as a bounded operator.

Following PrüB and Sohr (J. PRÜB and H. SOHR, 1990), the operator A is said to belong to the class $\mathrm{BIP}(X,\theta)$ for some $\theta \in [0,\pi)$ if :

(i) A is positive;

(ii) $N(A) = \{0\}$ and $R(A)$ dense in X;

(iii) $A^{is} \in B(X)$ $\forall\, s \in \mathbb{R}$ and there exists $M > 0$ such that $\|A^{is}\| \leq Me^{\theta|s|}$, $s \in \mathbb{R}$.

In the case where A is positive, $N(A) = \{0\}$ implies the density of $R(A)$ in X if X is a reflexive Banch space (a Hilbert space, for example).

It is proven in (J. PRÜB and H. SOHR, 1990), that if $A \in BIP(X,\theta)$ then A is of type (θ, M) for some $M \geq 1$. In Example A, we show in particular that the converse is not true even if the space X is a Hilbert space.

Example A. There exists an operator A in a Hilbert space which is of type (ω, M) for some $M > 1$ and for all $\omega \in (0,\pi)$ and such that the imaginary powers $A^{is}$are not bounded for all $s \in \mathbb{R} \setminus \{0\}$.

Remark. It is known (J. PRÜB and H. SOHR, 1990) that if an operator A in Hilbert space is of type *(ω, 1)* for some $\omega \in (0,\pi)$ (it is *m*-accretive), then $A \in \mathrm{BIP}(H, \pi/2)$.

Let A and B be two positive operators in a Banach space $(X, \|.\|)$. The operators A and B are called resolvent commutig if $(I + tA)^{-1}$ and $(I + sB)^{-1}$ commute for some t and $s > 0$ (equivalently for all t and $s > 0$).

Building upon results of Dore and Venni (G. Dore and A. Venni, 1987), and Sohr (J. PRÜB and H. SOHR, 1990) have proven that if $A_i \in \mathrm{BIP}(X,\theta_i), i = 1,2$ with $\theta_1 \neq \theta_2\,, \theta_1 + \theta_2 < \pi\,,$ are resolvent commuting and if X is a UMD-space, then $A_1 + A_2 \in \mathrm{BIP}(X,\theta)$ where $\theta = \max(\theta_1, \theta_2)$.

Da Prato and Grisvard (G. DA PRATO and P. GRISVARD, 1975) have proved that if A_i are of type $(\theta_i, M_i), i = 1,2,\ \theta_1 + \theta_2 < \pi\,,$ resolvent commuting (hence $A_1 + A_2$ closable) then the closure of $A_1 + A_2$ is of type (θ, M) with $\theta = \max(\theta_1, \theta_2)$ for some $M \geq 1$.

Therefore a natural question is to know whether the sum of two operators A and B satisfying the assumptions of Da Prato and Grisvard in a UMD-space is closed. In the Hilbert space, Da Prato and Grisvard (G. DA PRATO and P. GRISVARD, 1975) gave a sufficient condition for this to be the case, namely if the interpolation spaces $D_A(\theta, 2)$ and $D_{A^*}(\theta, 2)$ are equal for some $\theta \in (0,1)$ .Since $A + B$ is closed if and only if $I + A + B$ is closed, we may assume without loss of generality that $0 \in \rho(A)$ and $0 \in \rho(B)$. Under these assumptions Dore and Venni (G. Dore and A. Venni, 1987. p. 194), have shown that if the imaginary powers A is are uniformly bounded for $s \in [-1,1]$, then $A + B$ is closed .

Example B. There exists two resolvent commuting operators A and B in aHilbert space which are of type (ω, M) for some $M > 1$ and for every $\omega \in (0,\pi)$ such that $A + B$ is not closed.

Remarks. (i) It follows from Da Prato and Grisvard (G. DA PRATO and P. GRISVARD, 1975) that $D_A(\theta,2) \neq D_{A^*}(\theta,2)$ and $D_B(\theta,2) \neq D_{B^*}(\theta,2)$ for every $\theta \in (0,1)$.

(ii) It follows from Dore and Venni (G. Dore and A. Venni, 1987) that both A^{is} and B^{is} are not uniformly bounded on $[-1,1]$.

3. Tools

We recall the notion of spectral family of projections in a Hilbert space H (E. Berkson and T. A. Gillespie, 1987).

Definition. Aspectral family of projections in H is a uniformaly bounded projection–valued function $F: \mathbb{R} \to B(H)$ (the algebra of bounded linear operators in H) such that:

(i) F is right–continuous $\mathbb{R}$ in the strong operator topology,

(ii) F has a strong left–hand limit at each s ∈ ℝ ,

(iii) $F(s)\,F(t) = F(t)\,F(s) = F(s)$ for $s \leq t$,

(iv)$F(s) \to 0$ (resp. $F(s) \to I$) in the strong operator topology as $s \to -\infty$ (resp. as $s \to +\infty$).

If there is a compact interval $[a,b]$ such that $F(s) = 0$ for $s < a$ and $F(s) = I$ for $s \geq b$, then we say that F is concentrated on $[a,b]$. Following (E. Berkson and T. A. Gillespie, 1987), (H. R. DOWSON, 1987), if F is a spectral family concentrated on $[a,b$, each complex–valued function $f \in C[a,b] \cap BV[a,b]$ defines abounded operator A in H (BV stands for bounded variation) :

$$Ax = \int_{[a,b]} f(\lambda)\, dF(\lambda)x, \quad x \in H, \tag{3.1}$$

by means of convergence of Riemann-Stieltjes sums. Moreover the norm of A can be estimated by

$$\|A\| \leq |f(b)| + \big(|f(a)| + Var\big[f;[a,b]\big]\big).|||F|||, \tag{3.2}$$

Where

$$|||F||| := sup_{\lambda\in\mathbb{R}}\|F(\lambda)\|. \tag{3.3}$$

If F is concentrated on $[0,\infty)$ and $f \in C[0,\infty) \cap BV[0,\infty)$, then $s - \lim_{N\to\infty} \int_{[0,N]} f(\lambda)dF(\lambda)$ exists. This limit defines abounded operator A in H satisfying.

$$\|A\| \leq |f(\infty)| + (|f(0)| + Var[f;[0,\infty)]).|||F|||, \tag{3.4}$$

Where $|||F|||$ is defined by (3.3) and $f(\infty) = lim_{\lambda\to\infty} f(\lambda)$ which exists since $f \in BV[0,\infty)$.

If $f, g \in C[0,\infty) \cap BV[0,\infty)$ and

$$Ax = \int_{[0,\infty)} f(\lambda)dF(\lambda)x,$$

$$Bx = \int_{[0,\infty)} g(\lambda)dF(\lambda)x, \quad x \in H$$

then $(A+B)x = \int_{[0,\infty)} (f(\lambda) + g(\lambda))\, dF(\lambda)x$,

If moreover $f \in BV[0,\infty)$, then

$$ABx = BAx = \int_{[0,\infty)} f(\lambda)g(\lambda)dF(\lambda)x.$$

If $f(\lambda) \neq 0$, for every $\lambda \geq 0$ and $\lambda \mapsto f(\lambda)^{-1}$ belongs to $BV[0,\infty)$, then $0 \in \rho(A)$ and

$$A^{-1}x = \int_{[0,\infty)} f(\lambda)^{-1} dF(\lambda)x.$$

For the construction of a spectral family in $\ell^2(\mathbb{N})$ which is not spectral measure, we shall use, as in (E. Berkson and T. A. Gillespie, 1987), a conditional basis which can be found in Singer (I. SINGER, 1970). For the sake of completeness, we give it here explicitly.

Conditional Bases in $\ell^2(\mathbb{N})$. The sequences $\{f_n\}_{n\geq1}$ and $\{h_n\}_{n\geq1}$ in $\ell^2(\mathbb{N})$ defined by

$$f_{2n-1} = e_{2n-1} + \sum_{i=n}^{\infty} \alpha_{i-n+1} e_{2i},$$
$$f_{2n} = e_{2n}, \quad (n = 1,2,\dots) \tag{3.5}$$

$$h_{2n-1} = e_{2n-1},$$
$$h_{2n} = -\sum_{i=1}^{n} \alpha_{i-n+1} e_{2i-1} + e_{2n}, \quad (n = 1,2,\dots) \tag{3.6}$$

Where $\{e_n\}_{n\geq1}$ is the canonical basis of $\ell^2(\mathbb{N})$ and , $\alpha_n \geq 1$, $n = 1,2,\dots$, $\sum_{j=1}^{\infty} j\alpha_j^2 < \infty$, $\sum_{j=1}^{\infty} \alpha_j = +\infty$, (e.g., one can take $\alpha_n = 1/n\ log\ (n+1)$) are biorthogonal conditional bases of $\ell^2(\mathbb{N})$. Defining $P_n \in B(\ell^2(\mathbb{N}))$ by

$$P_n x = (x, h_n) f_n, \quad x \in \ell^2(\mathbb{N}),\ n = 1,2,\dots,$$

Where $(.,.)$ is the scalar product, then each P_n aprojection with $P_m\,P_n = 0$ for $m \neq n$ satisfying

$$\lim_{n\to\infty} \sum_{j=1}^{n} P_j\, x = x,\ x \in \ell^2(\mathbb{N}), \tag{3.7}$$

Moreover

$$sup \left\| \sum_{j=1}^{n} P_{2j} \right\| = \infty. \tag{3.8}$$

4. Example A

we construct an example of appositive operator A in a Hilbert space H such that imaginary powers A^{is} are not bounded for $s \in \mathbb{R} \setminus \{0\}$, although A is of type (ω, M) for some $M > 1$ and for every $\omega \in (0,\pi)$.

In order to do that, we construct the operator A on a Hilbert product.

Let $\{H_k, \|.\|_k\}_{k\in\mathbb{Z}}$ be a family of complex Hilbert spaces. Let $(H, \|.\|)$ be the Hilbert product.

$$H = \left(\prod_{k\in\mathbb{Z}} H_k\right)_2 = \left\{x = (x_k), x_k \in H_k,\ \|x\|^2 = \sum_{k\in\mathbb{Z}} \|x_k\|_k^2 < \infty\right\}.$$

The family $\{A_k\}_{k\in\mathbb{Z}}$ of bounded operators on H_k, defines the following closed densely defined operator A on H:

$$D(A) := \left\{x = (x_k), x_k \in H_k, \sum_{k\in\mathbb{Z}} \|A_k x_k\|_k^2 < \infty\right\}$$

$$(Ax)_k := A_k x_k, k \in \mathbb{Z} \, for \, x = (x_k) \in D(A). \quad (4.1)$$

Moreover A is bounded if and only if $sup_{k\in\mathbb{Z}}\|A_k\|_k < \infty$ and if this is the case $\|A\| = \sup_{k\in\mathbb{Z}}\|A_k\|$.

We say that family of positive operators $\{A_k\}_{k\in\mathbb{Z}}$ satisfies Property (P)if:

(i) $\sigma(A_k) \subset [0,\infty)$;

(ii) for every $\theta \in [0,\pi)$, there is $M(\theta)$ independent of k, such that $\|(I+zA_k)^{-1}\|_k \le M(\theta)$ for every $k \in Z$ and every $z \in \Sigma_\theta$.

We have

Lemma 4.1. Let $\{A_k\}_{k\in\mathbb{Z}}$ be a family of bounded positive operators on $A_k, k \in \mathbb{Z}$ satisfying Property (P) then there exists $M \ge 1$ such that the operator A defined by (4.1), is of type (ω, M) for every $\omega \in (0,\pi)$.

Moreover if $N(A) = \{0\}$, then for every $x = (x_k) \in D(A) \cap R(A)$, and $s \in \mathbb{R}\backslash\{0\}$, we have $x_k \in D(A_k) \cap R(A_k)$ and $(A^{is}x)_k = (A_k)^{is}x_k, k \in \mathbb{Z}$.

Proof. (i) Let $z \in \mathbb{C}\backslash(-\infty,0]$ and let $\theta = arg\, z$. Let $y = (y_k) \in H$. Since A satisfies Property (P), $-z^{-1} \notin \sigma(A_k)$ and there exists $x_k \in H_j, k \in \mathbb{Z}$ such that

$$(1+zA_k)_{x_k} = y_k, \qquad k \in \mathbb{Z}.$$

Since $\|x_k\| \le M(\theta)\|y_k\|_k$ we have $x = (x_k) \in D(A)\; and\; \|x\| \le M(\theta)\|y\|$. Moreover since $(I+zA_k) = \{0\}$, we have $N(I+zA) = \{0\}$, $-z^{-1} \in \rho(A)$, and $\|(1+zA)^{-1}\| \le M(\theta)$. This implies that A is of type (ω, M) with $M = M(0)$, for every $\omega = (0,\pi)$.

(ii) Assume $N(A) = \{0\}$, then $N(A_k) = \{0\}$ for every $k \in \mathbb{Z}$. Let $x = (x_k) \in D(A) \cap R(A)$. Then clearly, $x_k \in D(A_k) = H_k$. Since $x = Ay$ for some $y \in D(A)$, we have $x_k = A_k y_k$, hence $x_k \in R(A_k)$. Therefore $A^{is}x$ and $(A_k)^{is}x_k$ are well–defined by (2.1), for $s \in \mathbb{R}\backslash\{0\}$. Since $((I+tA)^{-1}x)_k = (I+tA_k)^{-1}x_k$, $t > 0$, $x = (x_k) \in H$, we obtain $(A^{is}x)_k = (A_k)^{is}x_k, k \in \mathbb{Z}$. This completes the proof of Lemma 4.1.

Next we construct a family of bounded positive operators $\{A_k\}_{k\in\mathbb{Z}}$ in $\ell^2(\mathbb{N})$, such that $0 \in \rho(A_k)$ and satisfying Properly (P). Notice that the imaginary powers A_k^{is}, $s \in \mathbb{R}$, are then bounded. We give a necessary condition for $sup_{k\in\mathbb{Z}}\|A_k^{is}\|$ to be finite for some $s \in \mathbb{R}\setminus\{0\}$.

Lemma 4.2. Let $\{f_n\}_{n\ge1}$ be a (Schauder) basis on $\ell^2(\mathbb{N})$ with corresponding projections $\{P_n\}_{n\ge1}$.

Let $F: \mathbb{R} \to B(\ell^2(\mathbb{N}))$ be the spectral family concentrated on $[0,1]$ defined by

$$F(\lambda) = 0 \qquad for \qquad \lambda < 1/2$$

$$F(\lambda) = \sum_{k=1}^{n} P_k \quad for\; \frac{n}{n+1} \le \lambda < \frac{n+1}{n+2}$$

$$for\;\; n = 1,2,\ldots$$

$$F(\lambda) = 1 \qquad for \qquad \lambda \ge 1.$$

Then for every $k \in \mathbb{Z}$ and every $x \in \ell^2(\mathbb{N})$

$A_k x = \int_{[0,1]} e^{k\lambda} dF(\lambda)x$ is well–defined

and

(i) The family of operators $\{A_k\}_{k\in\mathbb{Z}}$ satisfies Property (P) and $0 \in \rho(A_k)$, $k \in \mathbb{Z}$.

(ii) For every $s \in \mathbb{R}$, the imaginary power A_k^{is} is bounded and $A_k^{is}x = \int_{[0,1]} e^{isk\lambda} dF(\lambda)x$, $x \in \ell^2(\mathbb{N})$, $k \in \mathbb{Z}$. Moreover $A_k^{is}x = A_1^{iks}$.

(iii) If for some $s \in \mathbb{R}\backslash\{0\}$, $sup_{k\in\mathbb{Z}}\|A_k^{is}\| < \infty$ then the basis $\{f_n\}_{n\ge1}$is unconditional.

(iv)If the basis $\{f_n\}_{n\ge1}$ is unconditional then for all, $s \in \mathbb{R}$, $sup_{k\in\mathbb{Z}}\|A_k^{is}\| < \infty$.

Proof. (i) For every $k \in \mathbb{Z}$ the function $\lambda \mapsto exp\,\{k\lambda\}$ is continuous, bounded, increasing, hence of bounded variation on [0,1]. Therefore A_k is well-defined and bounded on $\ell^2(\mathbb{N})$ as well as $A_k^{-1}x$. Moreover $A_k = A_1^k$.

Let $z \in \mathbb{C}\backslash(\infty,0]$ and $\theta = arg\; z$. Then the function $\lambda \to a(\lambda;k,z) := (1+z\,exp(k\lambda))^{-1}$ is continuous, bounded, and of bounded variation on [0,1]. Indeed $\left|1+ze^{k\lambda}\right|^{-1} = \left|1+|z|e^{i\theta}e^{k\lambda}\right|^{-1}$, then $|a(\lambda;z,z)| \le m_1(\theta)$ where

$$m_1(\theta) \le \begin{cases} 1 & when\; 0 \le |\theta| \le \dfrac{\pi}{2} \\ \dfrac{1}{sin|\theta|} & when\; |\theta| > \dfrac{\pi}{2}\,. \end{cases}$$

Moreover

$$var_{\lambda\in[0,1]}[a(\lambda;k,z)] = \int_{[0,1]} \left|\frac{d}{d\lambda}(a(\lambda;k,z)\right| d\lambda$$

$$= \int_{[0,1]} \frac{|kz|e^{k\lambda}}{|a(\lambda;k,z)|^2} d\lambda$$

$$= \int_0^{|k|} \frac{|z|e^{(signk)\lambda}}{|a((sign\,k)\lambda,1,z)|^2} d\lambda$$

$$\le \int_0^{\infty} \frac{|z|e^{(signk)\lambda}}{|(1+|z|e^{i\theta}e^{(signk)\lambda}|^2}$$

$$\le \int_0^\infty \frac{dt}{|(1+te^{i\theta}|^2} = m_2(\theta) \qquad with$$

$$m_2(\theta) = \begin{cases} 1 & if \quad \theta = 0 \\ \dfrac{\theta}{sin\theta} & if\;\; 0 < |\theta| < \pi. \end{cases}$$

Let $M(\theta) = \big(m_1(\theta) + m_2(\theta)\big).\||F\||$. We observe that $M(-\theta) = M(\theta)$ and $M(\theta)$ increases on $0 \le \theta \le \pi$.

Therefore $-z^{-1} \in \rho(A_k)$ and $\|(1+zA_k)^{-1}\| \le M(\theta)$, which implies that the family $\{A_k\}_{k\in\mathbb{Z}}$ satisfies Property (P).

(ii) Let $b=(\lambda;k,s) := exp(isk\lambda)\, for\, \lambda \in [0,1], k\in\mathbb{Z}, and\ s\in\mathbb{R}$ then $|b(\lambda;k,s)| \le 1$ and

$$var_{\lambda\in[0,\infty]} b(\lambda;k,z) = \int_0^1 \left|\frac{db}{d\lambda}(\lambda;k,z)\right| d\lambda = |sk|.$$

Hence $\int_{[0,1]} e^{isk\lambda}\, dF(\lambda)$ defines a bounded operator $C_{k,s}$ in $\ell^2(\mathbb{N})$ for every $s\in\mathbb{R}$ and $k\in\mathbb{Z}$. For $x=(x_k)\in c_{00}$ (finite sequences in $\ell^2(\mathbb{N})$), we have

$C_{k,s}x = \sum_{l=-m}^{m} exp\,(iskl)P_l x$ for some $m\in\mathbb{N}$ depending on x.

By using the Dunford integral for the imaginary power $A_k^{is}x$, we obtain

$$A_k^{is}x = \frac{1}{2i\pi}\int_\Gamma \lambda^{is}(\lambda - A_k)^{-1}x d\lambda$$

$$= \frac{1}{2i\pi}\int_\Gamma \lambda^{is}\sum_{l=-m}^{m}(\lambda - exp\,(kl))^{-1}P_l x d\lambda$$

$$= \sum_{l=-m}^{m}\frac{1}{2i\pi}\int_\Gamma \lambda^{is}(\lambda - exp\,(kl))^{-1}P_l x d\lambda$$

$$= C_{k,s}x.$$

Since both A_k^{is} and $C_{k,s}$ are bounded on $\ell^2(\mathbb{N}))$ and c_{00} is dense in $\ell^2(\mathbb{N}))$, we have $C_{k,s} = A_k^{is}$. We also have $A_k^{is} = A_1^{iks}$.

(iii) If $sup_{k\in\mathbb{Z}}\|A_k^{is}\| < \infty$ for some $s\in\mathbb{R}\backslash\{0\}$ then $sup_{k\in\mathbb{Z}}\|A_1^{iks}\| < \infty$ and without loss of generality, we may assume $s>0$. We also have $A_1^{iks} = \left(A_1^{is}\right)^k$. By using a result of Nagy (B. SZ-NAGY, 1947), there exists an equivalent Hilbertian norm $|||\cdot|||$ on H such that $|||A_1^{iks}||| = 1$ for every $k\in\mathbb{Z}$. (Take, e.g., $|||x||| = lim_{n\to\infty}\|A_1^{isn}x\|^2)^{1/2}$ where Lim is a Banach limit in $\mathbb{N}$.) Then A_1^{is} is unitary in $(H, |||\cdot|||)$ and $\{f_n\}_{n\ge1}$ are eigenvectors corresponding to the eigenvalues

$$\mu_n = e^{is\,n/(n+1)}, \quad n = 1,2,\dots .$$

Then for $m,n > s/2\pi$, $m\ne n$, we have $\mu_m \ne \mu_n$. Therefore $\{f_n\}_{n\ge s/2\pi}$ is an orthogonal system in $(H,|||\cdot|||)$, hence $\{f_n\}_{n\ge1}$ is an unconditional basis in $(H,|||\cdot|||)$ and also in $(H,\|\cdot\|)$.

(iv)Suppose the basis $\{f_n\}_{n\ge1}$ is unconditional. By using a characterization of unconditional bases, there exists a constant $C>0$ such that $\|\sum_{i=1}^{n}\alpha_i f_i\| \le C\|\sum_{i=1}^{n}|\alpha_i| f_i\|$ for every $n\in\mathbb{N}$ and every finite scalar sequence $\{\alpha_i\}$.

For $x\in H_0$ (the linear dense subspace spanned by $\{f_n\}_{n\ge1}, k\in\mathbb{Z}, s\in\mathbb{R}$, we have

$$\|A_1^{iks}x\| = \sum_{n\ge1}|exp(isk\ \ n/(n+1))|P_n x$$

the sum is finite. Hence

$\|A_1^{iks}x\| \le C\|\sum_{n\ge1}|exp(isk\ \ n/(n+1))|P_n x\| = C\|x\|$. Then $\|A_1^{iks}x\| \le C$.

After these preparations, we can easily construct the operator A.

Construction of A. Let $H_k = \ell_2(\mathbb{N}), k\in\mathbb{Z}$, and let $\{f_n\}_{n\ge1}$ be a conditional basis of $\ell^2(\mathbb{N}))$, for example, the basis defined in (3.5). Define A_k like in Lemma 4.2, then for every $s\in\mathbb{R}\backslash\{0\}, \sup_{k\in\mathbb{Z}}\|A_k^{is}\| = \infty$. Then define the operator A like in Lemma 4.1. The operator A is of type (ω, M) for some $M\ge1$, and for every $\omega\in(0,\pi)$. Moreover for $s\in\mathbb{R}\backslash\{0\}$, A^{is} cannot be bounded, otherwise $sup_{k\in\mathbb{Z}}\|A_k^{is}\|$ would be finite. There for the operator A satisfies all the required properties.

5. Example B

In this section, we construct an example of two resolvent commuting, closed operators A and B, in a Hilbert space H such that A and B are of type (ω, M) for some $M>1$ and every $\omega\in(0,\pi)$, with $A+B$ not closed.0

Let $H=\ell^2(\mathbb{N})$, $\{f_n\}_{n\ge1}$, be a (Schauder) basis in $\ell^2(\mathbb{N})$, and $\{p_n\}_{n\ge1}$ be the associated projections.

We shall denote by H_0 the linear dense subspace spanned by $\{f_n\}_{n\ge1}$.

Let $F:\mathbb{R}\to B(H)$ be the spectral family defined by

$$F(\lambda) = 0 \quad for\, \lambda < 1$$

$F(\lambda) = \sum_{k=1}^{[\lambda]} P_k$,where $[\lambda]$ denotes the greatest integer $\le\lambda$.

We define $|||F||| = sup_{\lambda\ge0}\|F(\lambda)\| < \infty$.

Lemma 5.1. Let H, H_0, and F be as above. Let $h:[0,\infty)\to[1,\infty)$ be a continuous and increasing function. For any $x\in H_0$, let

$$T_0 x = \sum_{n=1}^{\infty} h(n)P_n x, \qquad (\,is\, finite). \quad (5.1)$$

Then, for every $\theta\in(-\pi,\pi)$, there exists $M(\theta)>0$ such that for every $z\in\Sigma_\theta$, $I+zT_0$ is a bijection in H_0 and

$$\|(I+zT_0)^{-1}x\| \le M(\theta)\|x\| \quad \forall\ \ x\in H_0. \quad (5.2)$$

Moreover T_0 is closable and its closure T is of type (ω, M) for some $M>1$ for every $\omega\in(0,\pi)$ and satisfies $0\in\rho(T)$.

Proof of Lemma 5.1 . (i) Proof of (5.2). For every $z\in\mathbb{C}\backslash(-\infty,0]$, we define

$S_0 x = \sum_{n=1}^{\infty}(1/(1+zh(n)))P_n x, x\in H_0$. We get $(I+zT_0)S_0 = S_0(I+zT_0) = I_{|H_0}$. The spectral representation of S_0 is given by

$$S_0 x = \int_{[0,\infty)} \frac{1}{1+zh(\lambda)} dF(\lambda)x, \qquad x\in H_0.$$

By using (3.4), we have

$$\|S_0x\| \le \left(\frac{1}{|1+zh(\infty)|} + \frac{1}{|1+zh(0)|}\, |||F||| + \underset{[0,\infty)}{Var}\left[\frac{1}{1+zh(.)}\right] . |||F|||\right)\|x\|,$$

for every $x \in H_0, h(\infty) = lim_{\lambda\to\infty} h(\lambda) = sup_{\lambda\ge 0} h(\lambda)$, which may be infinite.

$\underset{[0,\infty)}{Var}\left[\frac{1}{1+zh(.)}\right] \le \int_0^\infty \frac{dt}{|1+e^{i\theta}t|^2} < \infty$ with $z = |z|e^{i\theta}$.

Then we get (5.2).

(ii) Closure of T_0. It is known, see, e.g., (G. DA PRATO and P. GRISVARD, 1975), that (5.2) implies that T_0 closable and that its closure T satisfies the same inequality. For the sake of completeness, we prove that T_0 closable.

Let $x_n \in H_0$ be such that $x_n \to 0$ and $T_0x_n \to y$ for some $y \in H$. We have to prove $y = 0$. Let $v \in H_0$ then for $t > 0$, we have $\|x_n + tv\| \le M\|x_n + tv + tT_0(x_n + tv)\|$ and $\|tv\| \le M\|(t(v + y) + t^2T_0v)\|$ by taking the limit. Hence $\|v\| \le M\|(x + y + tT_0v)\|$ and $\|v\| \le M\|v + y\|$ by letting $t \downarrow 0$ for every $v \in H_0$. Since H_0 is dense in H, $y = 0$.

(iii) Type of T. From (5.2), we get $\|y\| \le M(\theta)\|(I + zT)y\|$ for every $y \in D(T)$ and $z \in \Sigma_\theta$, which implies that $I + zT$ is injective and that $R(I + zT)$ is closed, hence $R(I + zT) \supset \overline{H}_0 = H$. Therefore $z^{-1} \in \rho(T)$ and $\|(\mathrm{I} + \mathrm{zT})^{-1}\mathrm{x}\| \le \mathrm{M}(\theta)\|\mathrm{x}\|$ holds for every $x \in H$.

(iv) $0 \in \rho(\mathrm{T})$. Let $L_0x = \sum_{n=1}^{\infty}(1/h(n))P_nx \ \ \forall x \in H_0$. L_0 is the inverse of T_0 by using (3.4), we get

$$\|L_0x\| \le \left(\frac{1}{h(\infty)} + \left(2 - \frac{1}{h(\infty)}\right) |||F|||\right)\|x\|, \forall v \in \mathrm{H}_0.$$

Then L_0 is bounded and densily defined. This implies that the closure of L_0 is the inverse of T.

Next, we consider properties of two operators A_0 and B_0 of the form given by Lemma 5.1.

Lemma 5.2. Let f and g be two continuous, increasing functions from $[0,\infty)$ into $[1,\infty)$. Let A_0 and B_0 be the corresponding operators in H_0 defined by

$$A_0x = \sum_{n=1}^{\infty} f(n)P_nx \quad and$$

$$B_0x = \sum_{n=1}^{\infty} g(n)P_nx, \quad \forall \ x \in H_0 .$$

Let A and B be their closure in H.

Then, we have

(i) $\mathrm{A}_0(\mathrm{A}_0 + \mathrm{B}_0)^{-1} = (\mathrm{A}_0 + \mathrm{B}_0)^{-1}\mathrm{A}_0$ on H_0 ;

(ii) A and B are resolvent commuting;

(iii) $A + B$ is closable and $\overline{A+B} = \overline{A_0 + B_0}$.

Proof . (i) We have $A_0B_0x = (\Sigma_n f(n)P_n)\,(\Sigma_m g(m)P_mx) = \Sigma_n f(n)g(n)P_nx = B_0A_0x$ for every $x \in H_0$. Since $A_0 + B_0$ is a bijection on H_0, it follows that A_0 and $(\mathrm{A}_0 + \mathrm{B}_0)^{-1}$ commute.

(ii) As is well known it suffices to prove $(I + A)^{-1}(I + B)^{-1} = (I + B)^{-1}(I + A)^{-1}$. But this is a consequence of the commutativity of $(I + A_0)^{-1}$ and $(I + B_0)^{-1}$ on H_0 together with their boundedness.

(iii) First we prove that $A + B$ is closable. Let $x_n \in D(A) \cap D(B)$ be such that $x_n \to 0$ and $y_n: (A + B)x_n \to y$ with $y \in H$. Then

$$(I + A)^{-1}(I + B)^{-1}y_n = (I + A)^{-1}(I + B)^{-1}Bx_n + (I + B)^{-1}(I + A)^{-1}Ax_n$$
$$= (I + A)^{-1}[I - (I + B)^{-1}]x_n + (I + B)^{-1}[I - (I + A)^{-1}]x_n \to 0.$$

Hence $(I + A)^{-1}(I + B)^{-1}y = 0$, and $y = 0$.

Since the closure of $A_0 + B_0$ is contained in the closure of $A + B$, we only have to prove $\overline{A+B} \subset \overline{A_0 + B_0}$ or $A + B \subset \overline{A_0 + B_0}$. Let $x \in D(A) \cap D(B) = D(A + B)$. Then there are tow sequences $x_n, x'_n \to x$ and $A_0x_n \to Ax$ and $B_0x'_n \to Bx$.

Set $h_n = x'_n - x_n$. We have

$$x_n = (A_0 + B_0)^{-1}(A_0x_n + B_0x'_n) - B_0(A_0 + B_0)^{-1}h_n \qquad (5.3)$$

by using part (i). Since $(A_0 + B_0)^{-1}$ is bounded by Lemma 5.1, we obtain that the sequence $B_0(A_0 + B_0)^{-1}h_n$ converges to some $v \in H_0$. Moreover $(A_0 + B_0)^{-1}h_n \to 0$, then $v = 0$ since B_0 is closable by Lemma 5.1. Rewriting (5.3), we get

$$(A_0 + B_0)(x_n + B_0(A_0 + B_0)^{-1}h_n) = A_0x_n + B_0x'_n$$

which implies by passing to the limit

$x \in D\left(\overline{A_0 + B_0}\right)$ and $\left(\overline{A_0 + B_0}\right)x = Ax + Bx.$

Corollary 5.3. Let $\{f_j\}, \{g_j\}$ be two increasing continuous sequences of functions from $[0,\infty)$ into $[1,\infty)$. Let $\{\mathrm{A}_{0_j}\}$ and $\{\mathrm{B}_{0_j}\}$ be the corresponding operators in H_0 defined by

$$A_{0j}x = \sum_{j\ge 1}\sum_{n=1}^{\infty} f_j(n)p_nx$$

and

$$B_{0_j}x = \sum_{j\ge 1}\sum_{n=1}^{\infty} g_j(n)\,p_nx, \qquad \forall x \in H_0. \qquad (5.4)$$

Let $\{A_j\}$, $\{B_j\}$ be their closure in H then we have

$$A_{0j}\left(A_{0j} + B_{0j}\right)^{-1} = \left(A_{0j} + B_{0j}\right)^{-1} A_{0j}$$

on H_0.

A_j and B_j are resolvent commuting .

$A_j + B_j$ is closable and $\overline{A_j + B_j} = \overline{A_{0j} + B_{0j}}$.

Proof . Lemma 5.2 implies that

$$A_{0j}B_{0j}x = B_{0j}A_{0j}x$$

since x is total we have $A_{0j}B_{0j} = B_{0j}A_{0j}$. It follows that $A_{0j}(A_{0j}+B_{0j})^{-1} = (A_{0j}+B_{0j})^{-1}A_{0j}$, $(A_{0j}+B_{0j})A_{0j} = A_{0j}(A_{0j}+B_{0j})$, since $A_{0j}+B_{0j}$ is a bijection, then $A_{0j}^2 + B_{0j}A_{0j} = A_{0j}^2 + A_{0j}B_{0j}$, implies that $B_{0j}A_{0j} = A_{0j}B_{0j}$.

$$(1+A_j)^{-1}(1+B_j)^{-1} = (1+B_j)^{-1}(1+A_j)^{-1},$$
$$(1+B_j)^{-1}(1+A_j) = (1+A_j)(1+B_j)^{-1}$$
$$(1+A_j)(1+B_j) = (1+B_j)(1+A_j)$$
$$1+B_j+A_j+A_jB_j = 1+A_j+B_j+B_jA_j$$

hence $A_jB_j = B_jA_j$.

Follows directly from Lemma 5.2.

Now we give a Lemma which characterizes the closedness of $A+B$.

Lemma 5.4. Let the operators A and B be defined as in Lemma 5.2. Then $A+B$ is not closed if and only if there exists a sequence x_n in H_0 such that

$$\|x_n\| \le 1 \text{ and } \sup_{n\ge1} \|A_0(A_0+B_0)^{-1}x_n\| = \infty \quad (5.5)$$

Proof . (i) Let $E = D(A) \cap D(B)$. We define two norms on E:

$$\|x\|_1 := \|x\| + \|Ax\| + \|Bx\| \quad and$$
$$\|x\|_2 := \|x\| + \|(A+B)x\|, \quad x \in E.$$

Clearly $\|x\|_1 \ge \|x\|_2$ for $x \in E$. A and B are closed, E is complete with respect to the norm $\|.\|_1$. Moreover E is complete with respect to $\|.\|_2$ if and only if $A+B$ is closed. By using the open mapping theorem (for one implication), one has $A+B$ is closed if and only if there exists $C>0$ such that

$$\|x\|_1 \le C\|x\|_2 \quad \forall\ x \in E. \quad (5.6)$$

(ii) Let $x_n \in H_0$ be such that $\|x_n\| \le 1$ and $y_n = (A_0+B_0)^{-1}x_n$ with $Sup_{n\ge1}\|A_0y_n\| = +\infty$. Then (5.6) cannot hold. Indeed, we have

$$\begin{aligned}\|y_n\|_2 &= \|y_n\| + \|(A_0+B_0)y_n\| \\ &= \|(A_0+B_0)^{-1}x_n\| + \|x_n\| \\ &\le \|(A_0+B_0)^{-1}\| + 1\end{aligned}$$

and

$\|y_n\|_1 \ge \|A_0y_n\|$ which is unbounded.

Hence $A+B$ is not closed .

(iii) Assume $C_A = sup\{\|A_0(A_0+B_0)^{-1}y\|, \|y\| \le 1,\ y \in H_0\} < \infty$. By triangular inequality, there is $C_B > 0$ such that

$$\|B_0(A_0+B_0)^{-1}y\| \le C_B\|y\|, \quad \forall\, y \in H_0.$$

Then if $x = (A_0+B_0)y$, we have

$$\begin{aligned}\|y\|_1 &= \|y\| + \|A_0y\| + \|B_0y\| \\ &= \|y\| + \|A_0(A_0+B_0)^{-1}x\| + \|B_0(A_0+B_0)^{-1}x\| \\ &\le \|y\| + (C_A+C_B)\|x\| \le (1+C_A+C_B)\|y\|_2,\end{aligned}$$

$\forall\, y \in H_0$.

Then the norms $\|.\|_1$ and $\|.\|_2$ are equivalent on H_0. Observe that $H_0 = D(A_0+B_0)$. which is dense in $D(\overline{A_0+B_0})$ with respect to the norm $\|x\|_3 := \|x\| + \left\|\overline{(A_0+B_0)}x\right\|, x \in D(\overline{A_0+B_0})$. Notice that $E = D(A+B) \subset D(\overline{A_0+B_0}) = D(\overline{A+B})$.

Hence H_0 is dense on E with respect to $\|.\|_3$. For $x \in E$ there exists $x_n \in H_0$ such that $\|x - x_n\|_3 \to 0$ and $\|x\|_3 = lim_{n\to\infty}\|x_n\|_3 = lim_{n\to\infty}\|x_n\|_2 = \|x\|_2$, by using the continuity of $\|.\|_2$ on E. It follows that the norm $\|.\|_1$ and $\|.\|_2$ are equivalent on E.

Construction of the Example B. It is enough to choose A and B as in Lemma 5.1 and 5.2 such that condition (5.5) of Lemma 5.3 is satisfied, i.e., to find tow functions f and g as in Lemma 5.1 such that

$$Sup\left\{\left\|\sum_{n=1}^{\infty} \frac{f(n)}{f(n)+g(n)} P_n x\right\|, x \in H_0, \|x\| \le 1\right\} = \infty. \quad (5.7)$$

We show that this is possible .

First we choose for $\{f_n\}_{n\ge1}$ the conditional basis of example (3.5) which satisfies

$$\sup_{m\ge1}\left\|\sum_{n=1}^{m} P_{2n}\right\| = +\infty.$$

If we impose the following conditions on f and g,

$$\frac{f(n)}{f(n)+g(n)} = \begin{cases} 1/4 & for\ n\ odd \\ 3/4 & for\ n\ even \end{cases} \quad (5.8)$$

then

$$\sum_{n=1}^{2m} \big(f(n)/(f(n)+g(n))\big) P_n x = (1/4)\sum_{n=1}^{2m} P_n x + (1/2)\sum_{n=1}^{m} P_{2n} x,$$

which satisfies (5.7).

Finally, we give one possible choice of functions f and g satisfying the hypothesis of Lemma 5.1 and condition (5.8).

Set $h(t) = 1/2 + 1/4\cos(\pi t)$, $t \ge 0$.

We contract f and g by induction :

$f(0) = 3$ and $g(0) = 1$.

Suppose we know the functions between $[0,2n]$, $n = 0,1,2,\dots$ then we define for $t \in (2n, 2n+1]$

$$f(t) = f(2n) \quad \text{and} \quad g(t) = f(2n)\left(\frac{1}{h(t)} - 1\right)$$

and for $t \in (2n+1, 2n+2]$

$$f(t) = g(2n+1)\frac{h(t)}{1-h(t)} \text{ and } g(t) = g(2n+1).$$

Then, f, g are continuous on $[0,\infty)$, nondecreasing, not less than one with $f(t)/\big(f(t)+g(t)\big) = h(t)$.

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3/30/2014