Fixed Point Theorem And Its Application

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ABSTARCT: This paper examines the existence of fixed point and its application. An introduction to fixed point theorem and its applications to Linear equation are enumerated and proved. Since application usually involved space function, we give the Banach Space of the theorem. [Report and Opinion. 2009;1(1):17-34]. (ISSN: 1553-9873).

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INTRODUCTION
Definition 1

The importance of fixed-point theorem in Functional Analysis is due to its usefulness in the theory of ordinary differential equation. The existence of construction of a solution to a differential equation is often reduced to the location of a fixed point for an operator defined on a subset of a space of a function. We use fixed point theorem on many occasions to determine the existence of periodic solution for Functional Differential Equation when solution are already known to exist {1.2].

Perhaps as a result of the importance of fixed theory in Applied Mathematics and Functional Analysis, it has developed into area of independent research, where several areas of Mathematics such as Homology theory, Degree theory and Differential Geometry have come to play a very significant role

Various attempts have been made by researchers to study and locate existence of solution to a family of Mappings. Thus this study includes the investigation of fixed point theorem for mapping of a set into its power set in I relation to a single-valued mapping {3.4}.

Furthermore, the study of fixed point theorem has developed its own method and ideas as illustrated by Kick {5}

Theorems (1.1)

(1.1) Let (x, d) be a complete metric space. Let T:x $\rightarrow$7 X be a mapping in the real space. If there exist a number a<1 such that d(T(x), T(y) $\leq \alpha d$ (x,y)) for each x and y in X_j when the sequence of iterates $\{T^n(X)\}_{n-1}^{\infty}$ converges to point of T for any X £ x . { 4 }

Remark (1.1)

Remark (1.1): Mapping T: X$\rightarrow$7X satisfying the condition d(T(x), T(y) : $\leq \alpha d$ (x,y)) are called contraction mapping. Theorem (1.1) is involved in many of the existence and uniqueness proofs of Ordinary differential equations.

In order to generalize, the Contraction Mapping theorem to a wider class of function, we have the Brower fixed point theorem stated as follows.

Theorem (1.2): BROWER FLXED POINT THEOREM

Let U be a closed unit ball of any finite dimensional Euclidean spaces. Let T:U$\rightarrow$U be continuous, then T has a fixed point. Theorem (1.2): is weaker than theorem (1.1) because the sequence of iterates need not converge to a unique fixed point.

An important generalization of theorem (1.2) is the Letschetz Fixed Point theorem which states that given a finite simplicial complex K and a continuous function T:K$\rightarrow$k, it is possible to define a non-negative W(T) called the Lefchetz number of T (with the property that T$\neq$0) implies the existence of a fixed point of T.

Another important generalization of the Brower Fixed Point theorem is the Schander-Tychoner theorem which is stated as follows:

(1.3) Let K be a compact subset of a locally convex topological space X. If T is a continuous mapping from K into k, then T has a fixed point. Attempts at making Theorem (1.3) easier to apply in Functional Analysis leads to the following modification.

(1.4) Let K be a bounded, closed and convex subset of a Banach space X. Let T:K$\rightarrow$k be a compact mapping, then T has a fixed point in K.

We need to note that consideration of domains which are only bounded and convex, are example given by Vidossich. Lipschitzian mapping with Lipschitz constant I, may fail to have a fixed point, even under the additional assumption that the domain be compact in the weak-star topology.

This problem was however resolved by Kirk's Theorem which gives further condition ensuring the existence of a fixed point. We thus consider the following definitions before stating Kirks theorem;

Definition (1)

Let X be a Banach space and let D$\subset$X. A mapping D into X is said to be a non-expansive, if for each x,yεD, IIT(x)-T(y) II $\leq \|x-y\|$

Definition (2)

Let K$\subset$K, be non convex subset, K is said to have a normal structure, if for each convex subset H of K consist of one point H consists of a non-dimensional point, that is there is a point X_0 in H such that sup

$$\left[\|X_0 - X\| : x\varepsilon H\right] < \sup\left(\|x-y\| : y\varepsilon H\right)$$

Theorem (1.5)

Let X be a Banach space and K a weakly compact convex subset of X, and suppose K has a normal structure, then any non-expensive mapping T:K$\rightarrow$K has a fixed-point.

Non-expensive mapping have proved to be a great importance in the study of non linear operator, interest in such mappings stems from the fact that they are intimately connected with an important class of operators, the accretive operators, introduced by T. Kato and F.E. Browder in (1967).. Roughly speaking, a mapping T of a normal linear space into itself is accretive if the solution U(t, x) to the initial value problem.

$$\frac{\delta u(t)}{\delta t} + Tu(t) = 0$$,

where u(0) = x is non expansive in x for each t > 0.

2.1 Definition (3) (Contraction)

Let X=(x,d) be a complete metric space. A mapping T:X$\rightarrow$x is called a contraction on X if there exist a positive number, 0<α< 1, such that $\|Tx - Ty\|$ = $\delta(Tx,Ty) \leq \alpha\partial(x,y)$

<u>Definition (4)</u> (Complete metric space

A metric space m is called complete metric space if every Cauchy's sequence converges to a point in M.

2.2 BANACH FIXED POINT THEOREM

Contraction Mapping Principle

Let X = (x,d) be a complete metric space. If T:X$\rightarrow$x is a contraction, there exist X £ X, such that Tx=x, then T has precisely one fixed point.

Proof: We construct a sequence (X_n) and show that it is Cauchy, so that it converges in the complete space X and then we prove that its limit X is a fixed of T has no fixed points.

We construct a sequence of iterates

$$\begin{aligned}
&X_{n+1} = 1 + TX_n \\
&X_0 : X_1 = TX_0 \\
&X_1 : X_2 = TX_1 \\
&X_2 : X_3 = TX_2 \qquad\qquad (1.0) \\
&. \quad . \quad . \\
&. \quad . \quad . \\
&. \quad . \quad . \\
&X_n : \quad X_n = \quad Tx_n
\end{aligned}$$

$$\partial(Tx,Ty) \leq \alpha\partial(x,y) \quad (a < 1) \qquad (1.1)$$

Clearly, the set of equations (1.0) is a sequence of the images of X_o under repeated application of T.

To show that X_n is Cauchy.

By (1.0) and (1.1), we have

$$\begin{aligned}
&\partial(X_{n+l}, X_n) \quad = \partial\ (TX_m, TX_{m-l}) \\
&\alpha\partial\ (X_m, X_{m-1}) \\
&0 \leq \alpha\partial(TX_m\text{-}, TX_m\text{-}_2) \qquad\qquad (1.2) \\
&\leq \alpha^2\partial\ (X_{m_}l, X_{m-3}) \\
&\leq \alpha^m\partial\ (X_I, X_0)
\end{aligned}$$

By using triangle inequality which stated that for every x,y$\in$X.

$\partial(X_n, X_n) \leq \partial(X_m, X_{m+1}) + \partial(X_{m+l}, X_{m+2}) + \ldots\ldots + \partial(X_{0-1}, X_n)$
$\leq \left(\alpha^m + \alpha^{m+1} + \ldots + \alpha^{n-1}\right)\delta(X_0, X_1)$

By using the formula for the sum of a geometric progression where n>m

$$S_n = \frac{a(1-r)u}{1-r} = a^m \frac{1-a^{u-m}}{1-\alpha} \qquad (1.3)$$

Since 0< α <1 and also $1-\alpha^{n-m} < 1$

$$\partial\ (X_m, Xu) \leq \frac{\alpha^m}{1-\alpha}\delta(X_0, X_1),\ 0<\alpha<l. \qquad (1.4)$$

Therefore, $\partial(X_o, X_1)$ is fixed and shows that X_m is Cauchy.
Since X is complete, X_m converges, say $X_m \rightarrow X$, as $m \rightarrow \infty$
To show that this limit X is a fixed point of the mapping T.

Let, $Tx = x$ and $Ty = y$ (1.5)

$\partial(x, Tx) \leq \partial(x, x_m) + \partial(x_m, Tx) \leq \partial(x, x_m) + \alpha\partial(y_n Ty)$

Then,

Also, $\partial(y, Ty) \leq \partial(y, y_n) + \partial(y_n Ty)$

We conclude that $\partial(x, Tx) = 0$, so that x=Tx. These shows that X is a fixed point of T and is the only fixed point of T because from Tx=x and Ty=y, we obtain by (1.2)

$\partial\ (x, y) = \partial\ (Tx, Ty) \rightarrow \alpha\partial\ (x, y)$
$\Rightarrow \partial\ (x,y) = 0$, since $\alpha < 1$

Therefore, x=y and hence the solution is unique.

Theorem (2.3.0) (Contraction on ball)

Let T be a mapping a complete metric space X = (x,d) into itself. Suppose T is a contraction on closed by Y = $\left\{ x \big/ \partial(x, x_0) \leq r \right\}$

That is, T satisfies (1.2) and converges to $X \varepsilon Y$. This X is a fixed point of T and is the only fixed point of T in Y.

Proof: We merely have to show that X_m is as well as X lie in Y. We put m = 0 in (2.2.4), change n to m. and use equation (1.6) to get $\partial(x_0, x_m) < \frac{1}{1-\alpha}\partial(x_0, x_1) < r$

Hence, all X_m's are in Y

Also, $x, y\varepsilon$, since $X_m \rightarrow X$ and Y is closed.

2.0 APPLICATION OF FIXED POINT THEOREM

Banach's fixed point theorem has important application to iteration. Three important field of application of the Banach's Linear Space are:

(i). Linear algebraic equation
(ii). Ordinary differential equation
(iii). Integral equation

Here, we restrict ourselves to linear algebraic equation. and consider an

application to Linear equation.
We take the set X of all ordered n-turples of real numbers written as

$$x = (\varepsilon_1, \varepsilon_2, \varepsilon_3, ...\varepsilon n),\ y = (n_1..........n_n),\ z < (\partial_1 / ...., S)$$

On x, we define a metric d by

$$\partial (x, z) = \max / \varepsilon_i - \varepsilon_j / \qquad (1.6)$$

X = (x, d) is complete

On X, we define T:X $\rightarrow X$ by y = Tx = cx+b (1.7)

where C=(C_{jk}) is a fixed real nxn matrix and b ε X, a fixed vector. In this study, all vectors considered are column vectors, because of the usual conventional of matrix multiplication.
We now have:

$$n_j = \sum_{K=1}^{n} (jk^{\varepsilon} K + \beta j,\ j = 1,2,.............,n$$

where b = (βi)
Setting W = (w_j) = Tz. We-obtain (1.2) and (1.6)

$$\partial (y,w) = \delta (Tx, Tz) = \max/nj-wj/$$

$$= \max_j / \sum_{k-1}^{n} cjk \left(^{\varepsilon} K + ^{\infty} k \right) /$$

$$\leq \max_j / \varepsilon i - \varepsilon_j / \max_j \sum_{k=1}^{n} / Cjk /$$

$$= \partial(x,z)_j \max \sum_{k=1}^{n} / Cjk /$$

$$\partial(y,w) \leq \alpha \partial(x,z),$$

Therefore,

where $\alpha = \max_j \sum_{k=1}^{n} / Cjk$ (1.7)

<u>Theorem (2.5)</u> (Linear Equation)

If a system x=cx+b, c=(Cjk), be given as n linear equation in n unknown $\varepsilon_1, \varepsilon_2, \varepsilon_3, ...\varepsilon n$ (the component of x).

Satisfies $\sum_{k=1}^{n} / Cjk / < (j = 1,2,....., n)$ (1.8)

It has precisely one solution x. This solution can be obtained as the limit of the iteration sequence $\left(x^{(0)}, x^{(1)}, x^{(2)}, ...\right)$, where $x^{(0)}$ is arbitrary and
$X^{(m+1)}$ = cx(m) +, m = 10. 1 (1.9)

Error bounds are

$$\partial\left(x^m = x\right) \leq \frac{\alpha}{1-\alpha} \partial\left(x^{(m-1)}, x^m\right) \leq \frac{\alpha^m}{1-\alpha} \partial\left(x^{(0)}, x^{(1)}\right) \qquad (1.10)$$

Remarks

Condition (1.9) is sufficient for convergence. It is a row sum criteria because it involve row sum obtain by summarizing the absolute values of the elements in row C. if we replaced (1.2)

By other metrics, we would obtain other conditions. A system of non linear equations in n unknown is usually written as Ax = C. where A is an n-rowed square matrix. Many iterative methods for (1.9) with det $A \neq 0$, are such that one writes A=B-G with a suitable non-singular matrix B. then (1.10) becomes Bx=Gx+C or x = $^{e-1}$(G-x+C)
This suggest the iteration (1.9) where, C = $B^{-1}G$, b=. $B^{-1}C$. this is illustrated using the following by two standard methods:

i. The Jacobi, which is largely of theoretical interest.
ii. The Gauss-Seidel iteration, which is largely of use in Applied Mathematics.

(i) JACOBI ITERATION

This iteration methods is defined as

$$\varepsilon j(m+1) = \frac{1}{ajj}\left(\partial j - \sum_{k=1} ajk\varepsilon_k{}^{cm}\right), j = 1,2,\ldots\ldots\ldots\ldots\ldots, n\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

where C = (vj) in (2.5.3) and we assume ajj $\neq 0$ for j=1,2,…………n.

This iteration is suggested by solving the jth equation in (1.4) for ε. It is not difficult to verify that (1.5) can be written as this.

C= $-D^{-l}$(A-O), b=D^{-I}C.

where D = diag(ajj) is the matrix whose non-zero elements are those of the principal diagonal of A.

Condition (1.4) applied to C in (1.5) is sufficient for the convergence of the Jacobi iteration. Since C in (1.5) is relatively simple, we can express (1.4) directly in terms of the element of A.

The result is the row sum criteria for the Jacobi iteration

$$\sum_{\substack{K=1 \\ K\neq j}}^{n} / \frac{a_{jk}}{a_{jj}} / \leq 1,\ j = 1,..n. \qquad (1.6)$$

or

$$\sum_{\substack{K=1 \\ K\neq J}}^{n} / a_{jk} / \leq a_{jj},\ j = 1,2,...n \qquad (1.7)$$

This shows that convergence is guaranteed if the elements in the principal diagonal of A are sufficiently large.

(ii) GAUSS-SEIDEL ITERATION

This is a method of successive corrections in which at every instance all the latest known component are used. The method is defined by:

$$\varepsilon j^{(m+1)} = 1\left(Vj - \sum_{k=1}^{j-1} a_{jk}\,\varepsilon K^{(m+1)} - \sum_{k=j+1}^{n} a_{jk}\,\varepsilon k^{(m)}\right) \qquad (1.8)$$

where j=1,2, ... , n and we again assume $a_{jj} \neq 0$ for all j. We obtain a matrix form of (1.8)

above by writing A=-L+D-V
where D is as in the Jacobi iteration and L and *V*, are lower and upper triangular matrix respectively with principal diagonal elements all zero, the minus being a matter of convention and convenience.

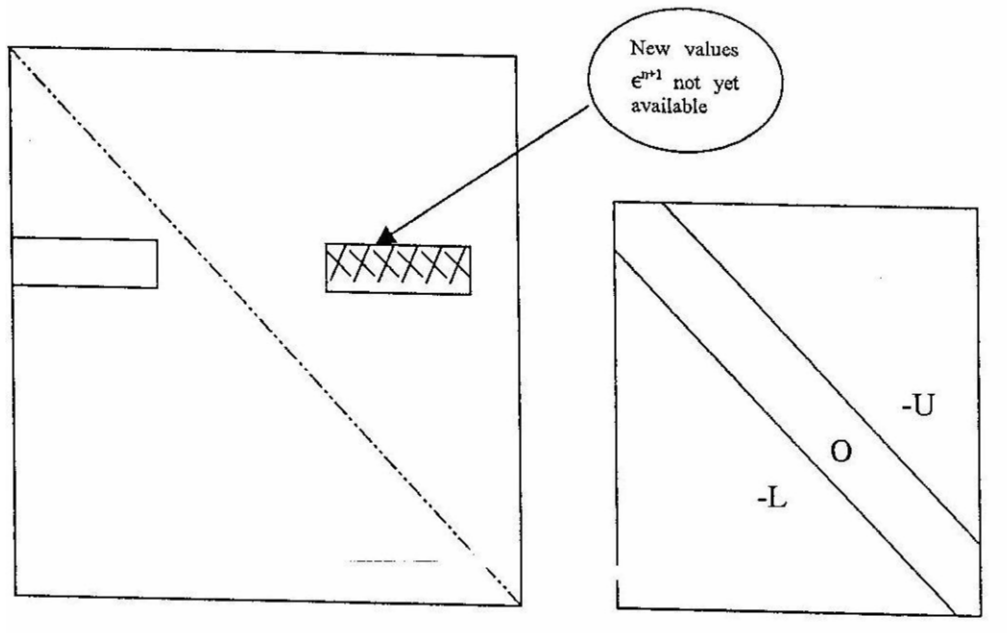

Fig (10) Decomposition of A

Also,

$$C= (D-L)^{-l}U \quad b=(D-L)^{-l}C \qquad (1.9)$$

Condition 1 applied to C in (1.9) sufficient for the convergence of the Gauss-Seidel iteration. Since C is complicated, the remaining practical problem is to get simpler conditions sufficient for the validity of (1.6).

Example (1.0)
Consider the system

$$z_l - 0.25z_2 - 0.25z_3 = 0.50$$
$$- 0.25z_1 + z_2 -0.25z_4 =0.50$$
$$- 0.25z_1 + z_2 -0.25z_4=C$$
$$- 0.25z_1 + g_3 -0.25z_4 =0.25$$
$$- 0.25z_2 - 0.25z_3 + z_4 =0.25$$

(a). Equations of these form arise in the numerical solution of partial differential equation.

(ii) Apply the Jacobi iteration, starting from X(0) with components (1, 1 , 1 , 1) the performing three stages.

Compare the approximating value with the exact values

$z_l=z_2=O.875$, $z_3=z_4=O.625$

(b). Apply the Gauss-Seidel iteration, performing the same tasks as in (a)

SOLUTION TECHNIQUE

$$\begin{bmatrix} 100 & -25 & -25 & 0 \\ -25 & 100 & 0 & -25 \\ -25 & -25 & -25 & 100 \\ 0 & 25 & -25 & 100 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{vmatrix} 50 \\ 50 \\ 25 \\ 25 \end{vmatrix}$$

$Ax_j = B_j$

$$x(0) = \begin{bmatrix} z_1(0) = 1 \\ z_2(0) = 1 \\ z_3(0) = 1 \\ z_4(0) = 1 \end{bmatrix}$$

(a) Using the formular $z_j\ (m + l) =$ $$\frac{1}{a_{jj}}\left(v_j - \sum_{\substack{k=1 \\ k\neq 1}}^{n} \partial jk - \varepsilon_k^{(m)}\right)$$
where V_j =diagonal elements, m=0, j=1

$$z_1^{(1)} = \frac{1}{a_{11}}\left(\partial_1 - \sum_{k=2}^{n} a_{1k}\varepsilon_k^{(0)}\right)$$

$$= \frac{1}{100}(50+50)$$

$$= 1.0000$$

When J=2, m=0

$$z_2(1) = \frac{1}{a_{22}}\left(v_j - \sum_{\substack{k=1 \\ k\neq 2}}^{n} \partial jk - \varepsilon_k^{(m)}\right)$$

$$= \frac{1}{100}\ (50+50)$$

$$= 1.0000$$

When m=0,J=3

$$z_3(1) = \frac{1}{a_{33}}\left(v_j - \sum_{\substack{k=1 \\ k\neq 3}}^{n} \partial jk - \varepsilon_k^{(m)}\right)$$

$$= \frac{1}{100}\ (50+25)$$

$$= 0.7500$$

When m=0,J=4

$$z_4(1) = \frac{1}{a_{44}}\left(v_j - \sum_{\substack{k=1\\k\neq 4}}^{4} \partial jk - \varepsilon_k{}^{(0)}\right)$$

$$= \frac{1}{100}\ (50+25)$$

$$= 0.7500$$

$$X^{(1)} = \begin{bmatrix} x_1{}^{(1)} \\ x_2{}^{(1)} \\ x_3{}^{(1)} \\ x_4{}^{(1)} \end{bmatrix} = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 0.7500 \\ 0.7500 \end{pmatrix}$$

From m=1, j =1
Using the fomular

$$z_1{}^{(2)} = \frac{1}{a_{jj}}\left(\gamma_j - \sum_{\substack{k=1\\k\neq j}}^{4} a_{jk}\varepsilon_k{}^{(1)}\right)$$

$$z_1{}^{(2)} = \frac{1}{a_{11}}\left(\partial_j - \sum_{\substack{k=1\\k\neq 4}}^{4} a_{1k}\varepsilon_k{}^{(0)}\right)$$

$$= \frac{1}{100}\left(50 - \left(a_{12}\varepsilon_1{}^{(1)} + a_{13}\varepsilon_3{}^{(1)}\right)\right)$$

$$= \frac{1}{100}\left(50 - \left(-25\times 1 + \left(-25\times 0.75\right)\right)\right)$$

$$= 0.9375$$

Using similar approach
Where m=1, J=2

$$z_2{}^{(2)} = \frac{1}{a_{22}}\left(\partial_2 - \sum_{\substack{k=1\\k\neq j}}^{4} a_{2k}\varepsilon_k{}^{(1)}\right)$$

$$=0.9375$$

In the same way,
When m=1, J=3

$$z_3{}^{(2)} = \frac{1}{a_{33}}\left(\partial_3 - \sum_{\substack{k=1\\k\neq 3}}^{4} a_{3k}\varepsilon_k{}^{(1)}\right)$$

$$=0.6875$$

Also, when m=1, J=4

$$z_4^{(2)} = \frac{1}{a_{44}}\left(\partial_4 - \sum_{\substack{k=1\\k\neq 4}}^{4} a_{4k}\varepsilon_k^{(1)}\right)$$

$$= \frac{1}{100}\left(25 + \frac{175}{4}\right)$$

$$=0.6875$$

Therefore,

$$x^{(2)} = \begin{bmatrix} z_1^{(2)} \\ z_2^{(2)} \\ z_3^{(2)} \\ z_4^{(2)} \end{bmatrix} = \begin{pmatrix} 0.9375 \\ 0.9375 \\ 0.6875 \\ 0.6875 \end{pmatrix}$$

When m=2, J=1

Using the following formula

$$Zj^{(m+1)} = \frac{1}{a_{jj}}\left(Yj - \sum_{\substack{k=1\\k\neq j}}^{4} a_{jk}\varepsilon_k^{(1)}\right)$$

$$z_1^{(3)} = \frac{1}{a_{11}}\left(\partial_4 - \sum_{\substack{k=1\\k\neq 1}}^{4} a_{1k}\varepsilon_k^{(2)}\right)$$

$$= \frac{90.625}{100}$$

When m=2, J=2

$$z_2^{(3)} = \frac{1}{a_{22}}\left(\partial_2 - \sum_{\substack{k=1\\k\neq 2}}^{4} a_{2k}\varepsilon_k^{(2)}\right)$$

$$= \frac{1}{100}(50 - (-23.4375 + (-17.1875))$$

$$= \frac{1}{100}(50 + 40.625)$$

$$= \frac{90.625}{100}$$

$$= 0.90625$$

$$z_3^{(3)} = \frac{1}{a_{33}}\left(\partial_3 - \sum_{\substack{k=1\\k\neq 3}}^{4} a_{3k}\varepsilon_k^{(2)}\right)$$

$$= \frac{1}{100}(25 - (-25 \times 0.9375) + 0 + (-25 \times 0.6875))$$

$= 0.65625$

When m=2, J=4

$$z_4^{(3)} = \frac{1}{a_{44}}\left(\partial_4 - \sum_{\substack{k=1 \\ k\neq 4}}^{4} a_{4k}\varepsilon_k^{(2)}\right)$$

$$= \frac{1}{100}\left(25 - \left(a_{41}g_1^{(2)} + a_{42}g_2^{(2)} + a_{43}g_3^{(2)}\right)\right)$$

$$= \frac{1}{100}\{25 - (0 + (-25\times 0.9375) + (-25\times 0.6875)\}$$

$$= 0.65625$$

Therefore,

$$X^{(3)} = \begin{bmatrix} z_1^{(3)} \\ z_2^{(3)} \\ z_3^{(3)} \\ z_4^{(3)} \end{bmatrix} = \begin{pmatrix} 0.90625 \\ 0.90625 \\ 0.65625 \\ 0.65625 \end{pmatrix}$$

To compare the approximation with the exact value $g_1 = g_2 = 0.825, g_3 = g_4 = 0.625$

Actual Error = Approximate-exact value

$= 0.90625-0.875$

$= 0.03125$ (for $z_1=z_2$)

Actual error for ($g_3=g_4$)

= Approximate value-exact value

$= 0.65625-0.625$

$=0.03125$

But X=

$$\begin{bmatrix} 0.875 \\ 0.625 \end{bmatrix}$$

The maximum value is 0.3125<1

The actual error is 0.3125, which indicates that the formula is still good for stability and convergence. To calculate error bound we have

$$\partial\left(X^{(3)}, X\right) = / X^{(3)} - X / \leq \frac{\alpha}{1-\alpha}\left(X^{(2)}, X^{(3)}\right)$$

$$\leq \frac{\alpha^m}{1-\alpha}\partial\left(X^{(0)}X^{(1)}\right) \qquad m = 3$$

$$= \frac{\alpha^3}{1-\alpha}\partial\left(X^{(0)}, X^{(1)}\right).$$

We pick largest value from $X^{(3)}$

i.e. $\alpha = 0.5$ from convergence criteria

$$\partial\left(X^{(0)}, X^{(1)}\right) = d\begin{bmatrix} 1-1 \\ 1-1 \\ 1-0.75 \\ 1-0.75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.25 \\ 0.25 \end{bmatrix}$$

But,

But 0.25 is the maximum.

Error bound now is

$$\partial\left(X^{(3)}, X\right) = \frac{(0.5)^3 .0.25}{1-0.5} = \frac{0.031625}{0.5} = \frac{0.031625}{0.5}$$

= 0.0625

Which is the same as the first method

Actual error = $|\text{Approximate - Exact}|$

= 0.90625-0.875 = 0.03125

= 0.65625-0.625 = 0.03125

Which shows that the formular is also accurate for the stability and convergence of the solution.

(b) Using the Formular,

$$zj^{(m+1)} = \frac{1}{a_{jj}}\left(Vj = \sum_{k=1}^{j-1} a_{jk}\varepsilon_k^{(m)} - \sum_{k=j+1}^{n} a_{jk}\varepsilon_k^{(m)}\right)$$

$$= \frac{1}{100}(50+50)$$

$$=1.0000$$

When m=0, J=2

$$Z_2^{(1)} = \frac{1}{a_{22}}\left(\partial_2 - \sum_{k=1}^{1} a_{2k}\varepsilon_K^{(1)} - \sum_{k=3}^{4} a_{2k}\varepsilon_k^{(0)}\right)$$

$$= \frac{1}{100}\left(50 + 25g_1^{\ 1} + 25\right)$$

$$= \frac{1}{100}(50+50)$$

$$= 1.0000$$

When m=0, J=3

$$Z_3^{(1)} = \frac{1}{a_{33}}\left(\partial_3 - \sum_{k=1}^{2} a_{3k}\varepsilon_K^{(1)} - \sum_{k=4}^{4} a_{3k}\varepsilon_k^{(0)}\right)$$

$$= \frac{1}{100}\left(25 + 25_1^{\ 1} + 25\right)$$

$$= \frac{75}{100} \qquad =0.7500$$

When m=0, J=4

$$Z_4^{(1)} = \frac{1}{a_{44}}\left(\partial_4 - \sum_{k=1}^{2} a_{4k}\varepsilon_K^{(1)} - \sum_{k=5}^{4} a_{4k}\varepsilon_k^{(0)}\right)$$

$$= \frac{1}{100}\left(25 - \left[a_{4k} g_1^{(1)} + a_{42} g_2^{(1)} + a_{43} g_3^{(1)}\right]\right)$$

$$= \frac{1}{100}\left(25 - (0 + (25 \times 1 + (-25 \times 0.7500)\ \right\}$$

$$= \frac{1}{100}(25 + 4375)$$

$$= 0.6875$$

$$X^{(3)} = \begin{bmatrix} g_1^{(1)} \\ g_2^{(1)} \\ g_3^{(1)} \\ g_4^{(1)} \end{bmatrix} = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 0.7500 \\ 0.6875 \end{pmatrix}$$

For m=1, J=1

$$Z_1^{(2)} = \frac{1}{a_{11}}\left(\partial_1 - \sum_{k=1}^{0} a_{1k}\varepsilon_K^{(2)} - \sum_{k=2}^{4} a_{1k}\varepsilon_k^{(1)}\right)$$

$$= \frac{1}{100}(50 + 25 + 18.75)$$

$$= \frac{1}{100}(93.75)$$

$$0.9375$$

$$Z_2^{(2)} = \frac{1}{100}\left(50 - \sum_{\substack{k=1 \\ 1}}^{0} a_{2k}\varepsilon_K^{(2)} - \sum_{\substack{k=3 \\ 4}}^{4} a_{2k}\varepsilon_k^{(1)}\right)$$

$$= \frac{1}{100}(90.63)$$

$$= 0.9063$$

When m=1, J=3

$$z_3^{(2)} = \frac{1}{100}\left(25 - \sum_{k=1}^{2} a_{3k}\varepsilon_K^{(2)} - \sum_{k=4}^{4} a_{3k}\varepsilon_k^{(1)}\right)$$

$$= \frac{1}{100}(65.63)$$

$$=0.6563$$

When m=1, J=4

$$z_4^{(2)} = \frac{1}{100}\left(25 - \sum_{k=1}^{3} a_{4k}\varepsilon_K^{(2)} - \sum_{k=5}^{4} a_{4k}\varepsilon_k^{(1)}\right)$$

$$z_4^{(2)} = \frac{1}{100}\left(25 - \sum_{k=1}^{3} a_{4k}\varepsilon_K^{(2)}\right)$$

$$=\frac{1}{100}(64.07)$$

$$= 64.07/100 \qquad =0.6407$$

$$X^{(2)}=\begin{bmatrix} z_1^{(2)} \\ z_2^{(2)} \\ z_3^{(2)} \\ z_4^{(2)} \end{bmatrix}=\begin{pmatrix} 0.9375 \\ 0.9063 \\ 0.6563 \\ 0.6407 \end{pmatrix}$$

When m=2, J=1

$$z_j^{(3)}=\frac{1}{a_{jj}}\left(\partial_j-\sum_{k=1}^{j-1}a_{jk}\varepsilon_K^{(3)}-\sum_{k=j+1}^{n}a_{jk}\varepsilon_k^{(2)}\right)$$

$$z_j^{(3)}=\frac{1}{100}\left(50-\sum_{k=1}^{0}a_{01}z_1^{(3)}+a_{13}z_3^{(2)}+a_{14}z_4^{(2)}\right)$$

$$=\frac{1}{100}(50--25\times 0.9063-25\times 0.6563+0)$$

$$=\frac{1}{100}(50+39.065)$$

$$=\frac{1}{100}(89.065)$$

$$=0.89065$$

When m=2, J=2

$$z_j^{(3)}=\frac{1}{a_{22}}\left(V_2-\sum_{k=1}^{1}a_{21}\varepsilon_K^{(3)}-\sum_{k=3}^{n}a_2\varepsilon_k^{(2)}\right)$$

$$z_2^{(3)}=\frac{1}{100}\left(50-\sum_{k=1}^{0}a_{21}z_1^{(3)}+a_{23}Z_3^{(2)}+a_{24}z_4^{(2)}\right)$$

$$=\frac{1}{100}(50-(-25\times 0.8907+25\times 0.6407)$$

$$=\frac{1}{100}(50+38.285)$$

$$=\frac{88.285}{100}$$

$$=0.8829$$

When m=2, J=3

$$Z_3^{(3)}=\frac{1}{a_{33}}\left(V_3-\sum_{k=1}^{2}a_3\varepsilon_K^{(3)}-\sum_{k=4}^{n}a_3\varepsilon_k^{(2)}\right)$$

$$=\frac{1}{100}(25-(-25\times 0.8907+25\times 0.6407))$$

$$=\frac{1}{100}(25-(-38.285)$$

$$=\frac{63.285}{100}$$

$$= 0.6329$$

When m=2, J=4

$$Z_4^{(3)} = \frac{1}{a_{44}}\left(V_4 - \sum_{k=1}^{3} a_4 \varepsilon_K^{(3)} - \sum_{k=5}^{n} a_4 \varepsilon_k^{(2)}\right)$$

$$= \frac{1}{100}\left(25 - a_{41} z_1^{(3)} + a_{42} z_2^{(2)} + a_{43} z_3^{(2)} + 0\right)$$

$$= \frac{1}{100}\left(25 - (0 - 25 \times 0.8829 - 25 \times 0.6329)\right)$$

$$= \frac{1}{100}\left(25 - (-37.895)\right)$$

$$= \frac{62.895}{100}$$

$$= 0.62895$$

$$X^{(3)} = \begin{bmatrix} z_1^{(3)} \\ z_2^{(3)} \\ z_3^{(3)} \\ z_4^{(3)} \end{bmatrix} = \begin{pmatrix} 0.8907 \\ 0.8829 \\ 0.6329 \\ 0.6290 \end{pmatrix}$$

Example (1.1)
Consider the system

$$5z_1 - z_2 = 7$$

$$-3z_1 + 10z_2 = 24$$

(a) Determine the Jacob iteration. Does C satisfies function starting with $X^{(0)} = 1, x_2(0) = 2$. Calculate $x^{(1)}$, $x^{(2)}$ and the error bounds for $x^{(2)}$.

Solution:

$$A = \begin{bmatrix} 5, & -1 \\ -3 & 10 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \end{bmatrix}$$

$$\text{Ax} = \text{d}$$

$$\sum_{k=1}^{n} / \frac{ajk}{ajj} / < \frac{3}{5} < 1, \frac{1}{10} < 1$$

$$X^{(0)} = \begin{bmatrix} 1, z_1^{(0)} \\ 1, z_2^{(0)} \end{bmatrix}$$

Using the Formular

$$= \quad zj^{(m+1)} = \frac{1}{ajj}\left(\gamma j - \sum_{k=1}^{n} a_{jk} \varepsilon_k\right)$$

When m=0, J=1

$$z_1^{(1)} = \frac{1}{a_{11}}\left(\partial_1 - \sum_{k=1}^{2} a_{1k} \varepsilon_k^{(0)}\right)$$

$$= \frac{1}{5}(7-(-1)\times 1)$$
$$= {}^{8}\!/_{5}$$

When m=0, j=2

$$z_2^{(0)} = \frac{1}{a_{22}}\left(\partial_2 - \sum_{k=1}^{2} - \sum_{\substack{k=1\\k\neq 1}}^{2} a_{2k}\,\varepsilon_k^{(0)}\right)$$
$$= \frac{1}{10}(24-(-3)\times 1)$$
$$= {}^{27}\!/_{10}$$
$$X^{(1)} = \begin{bmatrix} {}^{8}\!/_{5} \\ {}^{27}\!/_{10} \end{bmatrix}$$

When m=a, J=1

$$z_2^{(2)} = \frac{1}{a_{22}}\left(\partial_2 - \sum_{k=1}^{2} - \sum_{\substack{k=1\\k\neq 1}}^{2} a_{2k}\,\varepsilon_k^{(1)}\right)$$
$$= \frac{1}{10}\left(24+3\times {}^{8}\!/_{5}\right)$$
$$= {}^{144}\!/_{50} \qquad\qquad = {}^{70}\!/_{25}$$
$$X^{(2)} = \begin{bmatrix} {}^{97}\!/_{50} \\ {}^{72}\!/_{25} \end{bmatrix}$$

To calculate the error bound

$$\partial\left(x^{(2)},x\right) = /\,x^{(2)} - x\,/ < \frac{\alpha\partial}{1-\alpha}\left(x^{(1)},x^{(1)}\right)$$
$$\leq \frac{\alpha^2}{1-\alpha}\partial\left(x^{(0)},x^{(1)}\right)$$

$\alpha = {}^{3}\!/_{5}$

From convergence criteria

$$\partial\left(x^{(2)},x\right) \leq \frac{3/5}{1-3/5}\partial\left(x^{(2)},x^{(2)}\right) \leq \frac{3/5}{1-3/5}\partial\left(x^{(0)},x^{(1)}\right)$$
$$\max_j /\,x_i - y_i\,/$$

Now, $\partial\left(x^{(0)},x^{(1)}\right) = \max\left({}^{3}\!/_{5}, {}^{17}\!/_{10}\right)$

Now, $\partial\left(x^{(2)}, x\right) \leq \dfrac{\left(3/15\right)^2}{1-\left(3/5\right)}$

Error bounds

$$X^2 - X = \begin{bmatrix} 97/50 & -2 \\ 72/25 & -3 \end{bmatrix} = \begin{pmatrix} 3/50 \\ -3/25 \end{pmatrix}$$

NB: Absolute choice of higher value is $\left(3/25\right)$

When m=0,j=3

$$Z_3^{(1)} = \frac{1}{a_{33}}\left(\partial_3 - \sum_{\substack{k=1 \\ k\neq 3}}^{3} a_{3k}\,\varepsilon_k^{(0)}\right)$$

$$= \frac{1}{100}(25+50)$$

$$= \frac{75}{100} = 0.7500$$

When m=J=4

$$Z_4^{(1)} = \frac{1}{a_{44}}\left(\partial_4 - \sum_{\substack{k=1 \\ k\neq 4}}^{3} a_{4k}\,\varepsilon_k^{(0)}\right)$$

$$\frac{1}{100}(25+50)$$

$$= \frac{75}{100} = 0.7500$$

$$X^{(1)} = \begin{bmatrix} z_1^{(0)} \\ z_2^{(0)} \\ z_3^{(0)} \\ z_4^{(0)} \end{bmatrix} = \begin{pmatrix} 1.0000 \\ 1.0000 \\ 0.7500 \\ 0.7500 \end{pmatrix}$$

NON-EXPANSIVE MAPPING

(3.0)

Recall from the definition above that if x is a Banach space and DCX, then a mapping D into X is said to be non-expansive if for each x,y ε D //Tx-Ty// $\leq$ //x-y//.

Consider the iterative Scheme $X_{n+1} = Tx^n,\ x_0 \varepsilon X$ ....(3.0)

If T is non-expensive, that is if //Tx-Ty// $\leq$ //x-y// are there exist $X^* \varepsilon K, KCX$, such that $TX^* = X^*$.

Can we approximate X^* by a sequence of iterates of T?

The answer is NO

Reason

Consider $T\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

Let A= (aij) such that

//A//= $\underset{n}{Max}\left\{\sum_{j=1}^{n} / aij /^{2}\right)^{1/2}\Bigg\}$..................... *

* Is a norm, T is Non-expensive

To see this

//Tx-Ty//=// $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}^{(x,y)}$ //

= $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} //^{(x,y)}//$

Therefore, //Tx-Ty//≤//x-y// which shows that T is Non-expansive.

Now, Tx=x………………………..(3.1)

As = $\frac{1}{\sqrt{2}}\begin{bmatrix} 1, & -1 \\ 1, & 1 \end{bmatrix}^{(x)}$

From (3.0.1)

Tx-X=0

It implies that = $\frac{1}{\sqrt{2}}\left\{\begin{bmatrix} 1, & -1 \\ 1, & 1 \end{bmatrix} - \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}\right\} x = 0$........................(3.2)

Solving (3.2) implies x=0

This implies that the iterates cannot converse.

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