The Source of the Peano Axioms

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Abstract: The aim of this paper is to improve the conception of the natural numbers which is represented by the Peano axioms by introducing a non-arithmetical axiom. This way it can be demonstrated that if Ockham's razor is a correct principle then the terms 'zero', 'number' and 'immediate successor of' all spring from the same source. The multiplication of this source into three separate terms is upheld by the formal language but unnecessary for the clear mind.

Key words: Peano axioms; Ockham's razor; conception; non-arithmetical component; natural numbers

1. Introduction

Under consideration will be the five axioms related to the arithmetic of natural numbers (axiomatised by Giuseppe Peano in 1889) which are formulated with three undefined terms; 'zero', 'number' and 'immediate successor of', acquaintance with these terms being assumed [Nagel and Newman 1958 p.81]. These five axioms, hereafter called Peano axioms, can be stated as follows:

1. Zero is a number.
2. The immediate successor of a number is a number.
3. Zero is not the immediate successor of a number.
4. If the immediate successor of a number and the immediate successor of a number are the same then these numbers are also the same.
5. If a property belongs to zero, and also to the immediate successor of every number that has the property, then this property belongs to all numbers.

The following three statements will be examined:

I. Zero is not the immediate successor of a number.
II. Zero is the immediate successor of a number.
III. Zero is the immediate successor of not a number.

It is trivial that statement I is true because it follows immediately from axiom number 3. The falsehood of statement II is also clear because it is the negation of axiom number 3 and therefore contradicts this axiom. Statement III does not contradict the axioms as formulated here but this statement can neither be derived from these axioms, i.e., independent of these axioms. The meaningfulness of this statement, though, really depends on the assumed acquaintance with the undefined term 'immediate successor of'. Can statement III be accepted as sensible with respect to the conception of the natural numbers and if so, can statement III be accepted as a sentence with respect to the language of the formal system based upon the Peano axioms?

2. Discussion

The transformation of the conception of the arithmetic of natural numbers into a formal system can only be achieved by carving away all statements that do not obey the formality of this formal system. Apparently it is possible to find a meaningful statement (statement III) expressed in the vocabulary (calculus) of the formal system that does not obey the formation rules of this formal system, i.e., it cannot be formulated within arithmetic. This statement can be considered meaningful with respect to the conception of the natural numbers because either accepting statement III as true or as false results into a different representation of the conception of the natural numbers all be it in the non-arithmetical component. This means that either statement III or its negation is acceptable by the clear mind as sensible but neither statement III nor its negation is acceptable by the formal system as a sentence. In this case sense would appear for the formal system as non-sense.

In order to place the non-arithmetical component expressed by statement III or its negation onto the formal system one could add statement III or its negation to the list of axioms and this way extend the formal system. The system that is constructed this way can no longer be considered completely formal (unless the formation rules are adjusted) because it includes an informal sentence (statement III). This constructed system has the same vocabulary, the same rules of reasoning and an extended list of axioms. With respect to the formation rules it can be noted that this system has a formal language which reflects the conception of the arithmetical component of the natural numbers as well as an informal language which reflects the conception of the non-arithmetical component of the natural numbers. Hence, the extension of the formal system with statement III or its negation results into a non-formal system.

There are two possible ways of extending the formal system with respect to statement III; statement III can be considered true or false. If statement III is taken to be false then the negation of statement III 'Zero is *not* the immediate successor of not a number' can be added to the Peano axioms. If statement III is taken to be true then statement III itself can be added to the Peano axioms. What happens in the latter case, however, is quite interesting because this new axiom reveals that zero is an immediate successor, all be it of not a number. So, in this extended system all numbers are immediate successors and all immediate successors are numbers. This means that the term 'number' may be substituted for the term 'immediate successor'. If this substitution is realised then statement III expresses that 'zero is the immediate successor of not an immediate successor'. Hence the term 'zero' can be substituted for the term 'the immediate successor of not an immediate successor'. The total result of these two substitutions makes it possible to formulate the axioms of this extended system (in total five axioms instead of six because statement III has been encoded within the substitutions) as follows:

i. The immediate successor of not an immediate successor is an immediate successor.
ii. The immediate successor of an immediate successor is an immediate successor.
iii. The immediate successor of not an immediate successor is not the immediate successor of an immediate successor.
iv. If the immediate successor of an immediate successor and the immediate successor of an immediate successor are the same then these immediate successors are also the same.[1]
v. If a property belongs to the immediate successor of not an immediate successor, and also to the immediate successor of every immediate successor that has the property, then this property belongs to all immediate successors.[2]

Both these two possible extensions (statement III is true, statement III is false) represent the same conception with respect to the arithmetic of natural numbers which means that these two theories make exactly the same predictions with respect to the arithmetical component of the natural numbers, i.e., the theorems of both extensions that are the same can be formulated within arithmetic. So both theories predict the same observable facts. It is only with respect to the non-arithmetical component of the natural numbers that these two extensions represent a different conception which means that the theorems of both extensions that are different cannot be formulated within arithmetic. It now merits to be reminded of Ockham's razor (named after William of Ockham also spelled Occam) which is the principle that entities should not be multiplied unnecessarily, or, if two competing theories make exactly the same (observable) predictions then the simpler one (less assumptions, postulates) is better. If one is allowed to invoke upon Ockham's razor then of these two competing theories that make exactly the same observable predictions the non-formal system which is based upon the axioms i-v is considered better than the non-formal system which is based upon the Peano axioms and the negation of statement III because the latter has just like the Peano axioms an *assumed* acquaintance with three undefined terms whereas the former has an *assumed* acquaintance with only one undefined term; 'immediate successor'.

3. Conclusion

Peano's formal conception of the arithmetic of natural numbers is carved out of a non- formal system which represents a more complete conception of the natural numbers, that is, both the arithmetical component and the non-arithmetical component. If Ockham's razor is a correct principle then the non-formal system that is based upon the axioms i-v is considered better and reveals that the terms 'zero', 'number' and 'immediate successor of' all spring from the same source but differ in name. The multiplication of this source into three separate terms is upheld by the language of the formal system. The clear mind, however, which is undisturbed by the formation rules can conceive[3] this unnecessarily multiplication.

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11/30/2009

[1] This formulation simply reveals that distinct immediate successors have distinct immediate successors.

[2] It appears that this axiom has been derived from the following more complete formulation: 'If a property belongs to the immediate successor of not an immediate successor *that has not the property*, and also to the immediate successor of every immediate successor that has the property, then this property belongs to all immediate successors.'

[3] Statement III may also be conceived with the idea that zero arises from nothingness. Hence, zero immediately succeeds nothingness which is not a number.