Assessment of Homotopy Perturbation Method to Nonlinear Fifth-Order Boundary Value Problem

Abdolhosein Fereidoon [1], Hessameddin Yaghoobi [1]

[1.] Department of Mechanical Engineering, Faculty of Engineering, Semnan University, Semnan, Iran
Yaghoobi.Hessam@gmail.com

Abstract: Analytical solutions play a very important role in nonlinear boundry value problem. In this paper, He's homotopy perturbation method (HPM) has been applied to solve a nonlinear fifth-order boundary value problem. The concept of He's homotopy perturbation method are introduced briefly for applying this method for solving problem. The results of HPM as analytical solution are then compared with those derived from the exact solution in order to verify the accuracy of the proposed method. Just one iteration results in highly accurate solution. The results reveal that the HPM is very effective and convenient in predicting the solution of such problems, and it is predicted that HPM can have a found wide application in new engineering problems.
[Abdolhosein Fereidoon, Hessameddin Yaghoobi. **Assessment of Homotopy Perturbation Method to Nonlinear Fifth-Order Boundary Value Problem.** *Rep Opinion* 2015;7(2):58-60]. (ISSN: 1553-9873). http://www.sciencepub.net/report. 8

Keywords: Homotopy perturbation method (HPM); Nonlinear equation; Fifth-order; boundary value problem

1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. We know that most of engineering problems are nonlinear, and it is difficult to solve them analytically. Various powerful mathematical methods have been proposed for obtaining exact and approximate analytic solutions.

The homotopy perturbation method (HPM) was established by Ji-Huan He [1]. The method has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional equations. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. Considerable research work has recently been conducted in applying this method to a class of linear and non-linear equations.

The following fifth-order boundary value problems arise in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences [2-4]:

$$y^{(v)}(x)=f\left(x,y,y',y'',y''',y^{(iv)}\right) \tag{1}$$

with suitable boundary conditions:

$$\begin{aligned} &y(a)=A_1 \quad y'(a)=A_2 \quad y''(a)=A_3 \\ &y(b)=B_1 \quad y'(b)=B_2 \end{aligned} \tag{2}$$

where f is continuous function on $[a,b]$, and the parameters A_1, A_2, A_3, B_1, B_2 are real constants.

In this paper, we develop He's homotopy perturbation method by means of the computer for solving the system (1) with boundry condition (2). It is shown that this method is very easy to implement and it is more efficient than using sixth-degree B-spline functions [5] and variational iteration method [6]. An example is given to illustrate the performance of this method.

2. Basic Idea of Homotopy Perturbation Method

To explain this method, let us consider the following function:

$$S(u)-f(r)=0, r\in\Omega \tag{3}$$

With the boundary conditions of:

$$B\left(u,\frac{\partial u}{\partial n}\right)=0, r\in\Gamma \tag{4}$$

Where S is a general differential operator, $f(r)$ is a known analytic function; B is a boundary operator and Γ is the boundary of the domain Ω. The operator S can be generally divided into two operators, L and N, where L is a linear and N a nonlinear operator. Eq. (3) can be, therefore, written as follows:

$$L(u)+N(u)-f(r)=0 \tag{5}$$

Using the homotopy technique, we constructed a homotopy $v(r,p):\Omega\times[0,1]\rightarrow R$ which satisfies:

$$\begin{aligned} &H(v,p)=(1-p)[L(v)-L(u_0)]+ \\ &p[S(v)-f(r)]=0 \end{aligned} \tag{6}$$

Or

$$\begin{aligned} &H(v,p)=L(v)-L(u_0)+pL(u_0)+ \\ &p[N(v)-f(r)]=0 \end{aligned} \tag{7}$$

Where $p \in [0,1]$, is called homotopy parameter, and u_0 is an initial approximation for the solution of Eq. (3), which satisfies the boundary conditions. Obviously from Eqs. (6) and (7) we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{8}$$

$$H(v,1) = S(v) - f(r) = 0 \tag{9}$$

We can assume that the solution of (6) or (7) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{10}$$

Setting $p = 1$ results in the approximate solution of Eq. (3)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{11}$$

2.1. Implementation of HPM

To illustrate the implementation of this method, we consider the same example as in Wazwaz [7].

We consider the nonlinear BVP:

$$y^{(v)}(x) = e^{-x} y^2(x) \qquad 0 < x < 1 \tag{12}$$

with suitable boundary conditions:

$$\begin{array}{lll} y(0) = 1 & y'(0) = 1 & y''(0) = 1 \\ y(1) = e & y'(1) = e & \end{array} \tag{13}$$

To solve Eq. (12) with the initial condition (13), according to the homotopy perturbation, we construct the following homotopy:

$$L(v) = v^{(v)}(x) \tag{14a}$$

$$N(v) = -e^{-x} v^2(x) \tag{14b}$$

$$\begin{gathered} H(v,p) = v^{(v)}(x) - u_0^{(v)}(x) + p u_0^{(v)}(x) + \\ p\left[-e^{-x} v^2(x)\right] = 0 \end{gathered} \tag{14c}$$

Substituting $v = v_0 + pv_1 + p^2 v_2 + \dots$ into Eq. (14c) and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{d^5 v_0(x)}{dx^5} - \frac{d^5 u_0(x)}{dx^5} = 0 \tag{15a}$$

$$p^1 : \frac{d^5 v_1(x)}{dx^5} + \frac{d^5 u_0(x)}{dx^5} - e^{-x}(v_0(x))^2 = 0 \tag{15b}$$

With the following conditions:

$$\begin{array}{lll} v_0(0) = 1 & v_0'(0) = 1 & v_0''(0) = 1 \\ v_0(1) = e & v_0'(1) = e & \end{array} \tag{16a}$$

$$\begin{array}{lll} v_1(0) = 0 & v_1'(0) = 0 & v_1''(0) = 0 \\ v_1(1) = 0 & v_1'(1) = 0 & \end{array} \tag{16b}$$

With the effective initial approximation for v_0 from the conditions (16a), we obtain the solutions of Eq. (12):

$$\begin{gathered} v_0(x) = 0.0633\,x^4 + 0.1548\,x^3 + 0.5x^2 \\ + x + 1 \end{gathered} \tag{17a}$$

$$\begin{gathered} v_1(x) = (2730 + 2x^4 + 37x^3 \\ + 315x^2 + 1410\,x)e^{-x+1} + \\ (-7581 + -5.5x^4 - 102\,x^3 \\ - 870.5x^2 - 3906\,x)e^{-x} \\ + 21.05x^2 + 160 - 4.0466\,x^3 - 86.8x \end{gathered} \tag{17b}$$

Having $v_i, i = 0,1$, from Eq. (11), the solution $y(x)$ is as follows:

$$y(x) = v_0(x) + v_1(x) \tag{18}$$

The exact solution of the system (12) and (13) is

$$y(x) = e^x \tag{19}$$

3. Results and Discussion

In this section we present the results with HPM to show the efficiency of the method, described in the previous section for solving Eq. (12). From Fig. 1, it is clear that solution of homotopy perturbation method nearly equals exact solution. Table 1 exhibits a comparison between the errors obtained by using the homotopy perturbation method and by using variational iteration method [2] and sixth-degree B-spline method [14]. From this table, it is clear that an improvement is obtained by using the homotopy perturbation method. It is noticeable that the solution of HPM is from just one iteration. Higher accuracy can be obtained from higher iterations.

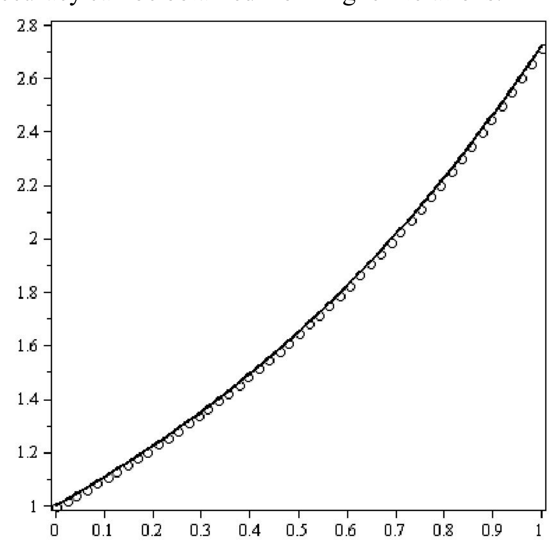

Figure 1. Comparison of the exact solution (continuous line) with the HPM solution (circle line).

Table 1. Error estimates (Error = exact solution - analytical or numerical solutions)

x	Exact solution y(x)	Error (HPM)	Error [6]	Error [5]
0	1.0000	0.0000	0.0000	0.0000
0.1	1.1052	-1.9E-6	0.0000	-7.0E-4
0.2	1.2214	-2.0E-5	1.0E-5	-7.2E-4
0.3	1.3499	-5.4E-5	1.0E-5	4.1E-4
0.4	1.4918	-9.9E-5	1.0E-4	4.6E-4
0.5	1.6487	-1.4E-4	3.2E-4	4.7E-4
0.6	1.8221	-1.6E-4	3.6E-4	4.8E-4
0.7	2.0138	-1.3E-4	-1.4E-4	3.9E-4
0.8	2.2255	-1.1E-4	-3.1E-4	3.1E-4
0.9	2.4596	-4.8E-5	-5.8E-4	1.6E-4
1	2.7183	-1.4E-5	-9.9E-5	0.0000

4. Conclusion

He's homotopy perturbation method (HPM) has been successfully utilized to derive approximate explicit analytical solutions for nonlinear fifth order boundry value problem. The results show that this perturbation scheme provides excellent approximations to the solution of this nonlinear equation with high accuracy and avoids linearization and physically unrealistic assumptions. This new method accelerated the convergence to the solutions. the results show that when parameter of x increases, the error of HPM is lower than two methods and this method is stable for higher x in this problem.

Acknowledgements:

The authors gratefully acknowledge the support of the department of mechanical engineering and the talented office of Semnan University for funding the current research grant.

2/25/2015

Corresponding Author:
Hessameddin Yaghoobi
Department of Mechanical Engineering,
Faculty of Engineering,
Semnan University, Semnan, Iran
E-mail: Yaghoobi.Hessam@gmail.com

References

1. H. He, Homotopy Perturbation Technique, Comput. Methods Appl. Mech. Eng. 178 (1999) 257–262.
2. H.N. Caglar, S.H. Caglar, E.E. Twizell, The numerical solution of fifth order boundary-value problems with sixth degree B-spline functions, Appl. Math. Lett. 12 (1999) 25-30.
3. A.R. Davies, A. Karageoghis, T.N. Phillips, Spectral Glarkien methods for the primary two-point boundary-value problems in modeling viscoelastic flows, Internat. J. Numer. Methods Engrg. 16 (1988) 647-662.
4. A. Karageoghis, T.N. Phillips, A.R. Davies, S pectral collocation methods for the primary two-point boundary-value problems in modeling viscoelastic flows, Internat. J. Numer. Methods Engrg. 26 (1998) 805-813.
5. H.N. Caglar, S.H. Caglar, E.E. Twizell, The numerical solution of fifth order boundary-value problems with sixth degree B-spline functions, Appl. Math. Lett. 12 (1999) 25-30.
6. J. Zhang, The numerical solution of fifth-order boundary value problems by the variational iteration method, Computers and Mathematics with Applications 58 (2009) 2347-2350.
7. A.M. Wazwaz, The numerical solution of fifth-order boundary-value problems by Adomian decomposition, J. Comput. Appl. Math. 136 (2001) 259-270.