Locally Planes Riemnnian Banach Manifolds

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Abstract: Theory of differentiable infinite dimensional manifolds [1-11] evolved considerably over the last thirty years. The necessary and sufficient condition for a Riemannian Banach manifold to be a locally plane space will be established. Also, in this work we proved that a Riemannian Banach manifold of constant sectional curvature is a locally plane space. MS classification: 53C40.

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1. Notation and definitions:

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Let M be a Riemannian Banach manifold of class $C^r(r \geq 3,\infty),$ modeled on a Banach space E [2].

The symmetric bilinear positive definite continuous functional $f \in L_2(E;IR)$ is said to be strongly non-singular [2], if f associates a mapping $f^* : x \in E \to f_x^* = f(x,.) \in L(E;IR) = E^*$ which is bijective.

Also, let $\bar{g}$ be the metric tensor on the space M , of class C^{r-1}. we assume that $\bar{g}$ is strong non-singular [2].

By $\nabla \bar{g}_{\bar{x}}$, we denote the covariant differentiation of the tensor $\bar{g}$ at the point $\bar{x} \in M.$ Finally, by $D\bar{g}_{\bar{x}}$ we mean the Frechet derivative of the metric $\bar{g}$.

2. Locally plane Riemannian Banach manifolds.

Since M is a Riemannian manifold, then on M there exists a unique torsion-free connection $\bar{\Gamma}$ [2] of class C^{r-2}, such that: $\nabla \bar{g} = 0.$

Definition (2.1) [2]: A Riemannian Banach manifold M is called locally plane space, if for all $\bar{x} \in M$, there exists a chart $c = (U, \Phi, E)$ at the point $\bar{x}$, such that $\Gamma_x \equiv 0,$ for all $x = \Phi(\bar{x}) \in \Phi(U) \subset E.$ Where x and Γ are the models of the point $\bar{x}$ and the connection $\bar{\Gamma}$ with respect to the chart c, respectively.

Lemma (2.1): The metric tensor $\bar{g}$ of a Riemannian Banach manifold M, which is locally plane, is a constant tensor field [4].

Now, we assume that $N \subset M$ is a submanifold of M of the same class[1]. Let $\bar{i} : \bar{x} \in N \to \bar{i}(\bar{x}) = \bar{x} \in M,$ be the inclusion map. Let $c = (U, \Phi, E)$ be a chart at the point $\bar{x} \in N \subset M$ on the space M , and $d = (V, \Psi, F)$ is a chart at the point $\bar{x} \in N \subset M$ on the space N .

If $Z=\Phi(\bar{x})$ and $P=\Psi(\bar{x})$ are the models of the point $\bar{x}$ with respect to the charts c and d, respectively. Also, if i is the model of the mapping $\bar{i}$ with respect to the charts c and d, then we have that:

$$i: P=\Psi(\bar{x})\in\Psi(V)\subset F\rightarrow i(P)=Z=\Phi(\bar{x})\in\Phi(U)\subset E.$$

This equation is called the local equation of the submanifold N in a neighbourhood of the point $\bar{x}\in N$ with respect to the charts c and d.

Now, since $(M,\bar{g})$ is a Riemannian manifold, then $N\subset M$, will be a Riemannian submanifold of M with respect to the induced metric tensor $\bar{g}^{1}$ such that [2]:

$$\bar{g}^{1}_{\bar{x}}(\bar{x}_1,\bar{x}_2)=\bar{g}_{\bar{i}(\bar{x})}(T_x\bar{i}(\bar{x}_1),T_{\bar{x}}\bar{i}(\bar{x}_2)), \tag{2.1}$$

for all $\bar{x}\in N,\ \bar{X}_1,\bar{X}_2\in T_{\bar{x}}N$ (the tangent space of N at the point $\bar{x}$). Also, we have that $T_{\bar{x}}\bar{i}: T_{\bar{x}}N\rightarrow T_{\bar{x}}M,$ is the tangent map of the map $\bar{i}$ at the point $\bar{x}$ [1].

Similarly, we assume that the metric $\bar{g}^{1}$ is strong non-singular. If X_1 and X_2 are the models of the vectors $\bar{X}_1$ and $\bar{X}_2$ with respect to the chart d, then the models of these vectors with respect to the chart c will be: $Y_1=Di_p(X_1), Y_2=Di_P(X_2),$ respectively. Hence, the local form of the equation (2.1) takes the form:

$$\overset{1}{g}_p(X_1,X_2)=g_{i(p)}(Di_p(X_1),Di_p(X_2)). \tag{2.2}$$

Also, with respect to the Riemannian submanifold N, there exists α unique torsion-free connection $\bar{\Gamma}^1$, such that[2]:

$$\bar{\nabla}^1\bar{g}^1=0$$

We assume that Γ and Γ^1 are the models of the connections $\bar{\Gamma}$ and $\bar{\Gamma}^{1}$ with respect to the charts c and d, respectively.

In [3], the first derivative equation of the submanifold N is established in the form:

$$\overset{1,2}{\nabla}Di_p(X_1,X_2)=n_p(A_p(X_1,X_2)), \tag{2.3}$$

where, $n: x=\psi(\bar{x})\in\psi(V)\subset F\rightarrow n_x\in L(W;F_x^{\perp})$ is an isomorphism of class C^{r-1} and $F_x^{\perp}$ is the orthogonal complement of the space F at the point $x\in F$.

Also, the space W is isomorphic to the space $(T_{\bar{x}}N)^{\perp}$. Finally, $\overset{1,2}{\nabla}$ is the mixed covariant differential operator defined on the tensors of the space N, and $A_p\in L_2(F;W)$ is the second essential form of the space N.

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In [4], it is proved that there exists a chart $C=(U,\Phi,E)$ at the point $\bar{x}\in N$, such that the relation between the metric tensor $\bar{g}$ and the connection $\bar{\Gamma}$ is given in the form:

$$g_x(\Gamma_x(X_1,X_2),X_3)=\frac{1}{2}[Dg_x(X_2;X_1,X_3)+Dg_x(X_1;X_2,X_3)-Dg_x(X_3;X_1,X_2)],$$

For all $x=\Phi(\bar{x})\in\Phi(U)\subset E, X_1,X_2,X_3\in E$. Where g_x and Γ_x are the models of the metric tensor $\bar{g}_{\bar{x}}$ and the linear connection $\bar{\Gamma}_{\bar{x}}$ with respect to the chart C in the above relation respectively.

Now, assume that $\bar{g}'$, is another Riemannian metric function on the space M.

Definition (2.2) [2] : The two functions $\bar{g}$ and $\bar{g}'$ are conformal, if there exists a mapping $\mu:\bar{x}\in M\rightarrow \mu(\bar{x})\in IR,$ such that:

$$\bar{g}'_{\bar{x}}=\mu(\bar{x}).\bar{g}_{\bar{x}}, \tag{2.5}$$

for all $\bar{x}\in M.$

Definition (2.3) [4]: The Riemannian Banach manifold $(M,\bar{g})$ is called locally plane, if there exists on M a locally plane metric $\bar{g}'$, conformal to $\bar{g}$ such that:

$\bar{g}'_{\bar{x}}=e^{\lambda(\bar{x})}.\bar{g}_{\bar{x}},\ \bar{R}'_{\bar{x}}\equiv 0,$ for all $\bar{x}\in M$, where $\bar{R}_{\bar{x}}'$ is the curvature tensor of the space $(M,\bar{g}')$ at the point $\bar{x}$.

Lemma (2.2) [2] : Let H be a vector space with a bilinear and strong non-singular operator g [2], such that $\dim\ H>3.$ Then, in the space H, we have that:

i- There exist two arbitrary vectors S,W and a vector Z, which is linearly independent with them;

ii- There exists a vector X, perpendicular to the vectors S,W and Z with respect to the metric $\bar{g}$ such that X is linearly independent of these vectors.

Now, we consider the following theorem:

Theorem (2.1): The necessary and sufficient condition for a Riemannian Banach manifold M with a strong non-singular metric $\bar{g}$, to be locally plane, is to find a symmetric tensor $\bar{P}_{\bar{x}}(\bar{X},\bar{Y})$ of type (0,2) on the space M, such that the following conditions are satisfied:

$$R_{\bar{x}}(\bar{X},\bar{Y},\bar{Z})=\bar{P}_{\bar{x}}(\bar{Y},\bar{X}).\bar{Z}+\bar{g}_{\bar{x}}(\bar{X},\bar{Y})\bar{\delta}(\bar{Z}), \tag{2.6}$$

$$\nabla\bar{p}_{\bar{x}}(\bar{X};\bar{Y},\bar{Z})=0, \tag{2.7}$$

Where $\bar{\delta}(\bar{x})$ is a tensor of type (1,1), is a solution of the equation:

$$\overline{P}_{\bar{x}}(\bar{X},\bar{Y}) = g_{\bar{x}}(\bar{Y},\bar{\delta}(\bar{X})) \tag{2.8}$$

Furthermore, when $\dim M > 3,$ we can show that, the condition (2.7) is a direct result, of the condition (2.6).

Remark (2.1) In equations (2.6) and (2.7), there exists an alternation with respect to the underlined vectors, that does not involve division by 2. This convention will be used henceforth.

Proof: It is sufficient to prove this theorem, locally, with respect to an arbitrary chart.

Necessity: We assume that $c = (U, \Phi, E)$ is a chart at the point $\bar{x} \in M$ such that g, Γ and R are the models of $\bar{g}, \bar{\Gamma}$ and , $\overline{R}$ respectively with respect to this chart.

Also, assuming that, the Riemannian Banach manifold $(M, \bar{g})$ is locally plane. Then there exists a locally plane metric $\bar{g}'$, conformal to $\bar{g},$ such that:

$$g'_x = e^{\lambda(x)} . g_x \ , \ R'_x \equiv 0, \tag{2.9}$$

for all $x = \Phi(\bar{x}) \in \Phi(U) \subset E.$ Therefore, we obtain:

$$g'_x(X, \Gamma_x'(Y,Z)) = \frac{1}{2}[Dg_x'(Y;X,Z) + Dg_x'(Z;Y,Y) - Dg_x'(X;Y,Z)].$$

Applying (2.9) in this last equation, yields:

$$\begin{aligned} g'_x(X, \Gamma'_x(Y,Z)) &= \frac{1}{2}[e^{\lambda(x)} Dg_x(Y;X,Z) + e^{\lambda(x)} D\lambda_x(Y) g_x(X,Z) + \\ & e^{\lambda(x)} Dg_x(Z;Y,X) + e^{\lambda(x)} D\lambda_x(Z).\ g_x(Y,X) - \\ & e^{\lambda(x)} Dg_x(X;Y,Z) - e^{\lambda(x)} D\lambda_x(X).\ g_x(Y,Z). \\ &= e^{\lambda(x)}[2g_x(X, \Gamma_x(Y,Z)) + \frac{1}{2}(D\lambda_x(Y).g_x(X,Z) + \\ & D\lambda_x(Z).g_x(X,Y) - D\lambda_x(X).g_x(Y,Z))]. \end{aligned} \tag{2.10}$$

But, from equation (2.9), we have:

$$g'_x(X, \Gamma'_x(Y,Z)) = e^{\lambda(x)} .g_x(X, \Gamma'_x(Y,Z)). \tag{2.11}$$

Using equations (2.10) and (2.11), we get:

$$g_x(X, \Gamma'_x(Y,Z)) = g_x(X, \Gamma_x(Y,Z)) + \frac{1}{2}[D\lambda_x(Y).g_x(X,Z) + D\lambda_x(Z).g_x(X,Y) \quad D\lambda_x(X).g_x(Y,Z)]. \tag{2.12}$$

Now, the function $D\lambda_x : X \in E \to D\lambda_x(X) \in IR$ is linear and continuous [1]. This means that $D\lambda_x \in L(E;IR) = E^*,$ where E^* is the dual space of the space $E.$ Hence, taking into account that the metric $\bar{g}$ is strong non-singular, then there exists a vector $B_X \in E$ such that:

$$g_x(X, B_x) = D\lambda_x(X), \tag{2.13}$$

for all $X \in E.$

Using equation (2.13) into equation (2.12), we get:

$$g_x(X,\Gamma_x'(Y,Z)) = g_x(X,\Gamma_x(Y,Z)) +$$
$$\frac{1}{2}[D\lambda_x(Y).g_x(X,Z) + D\lambda(Z).g_x(X,Y) - g_x(X,B_x).g_x(Y,Z)] =$$
$$g_x(X,\Gamma_x(Y,Z) + \frac{1}{2}[D\lambda_x(Y).\, Z + D\lambda_x(Z).\, Y - g_x(Y,Z).B_x].$$

Since, g is non-singular, we get:

$$\Gamma'_x(Y,Z) = \Gamma_x(Y,Z) + \frac{1}{2}[D\lambda_x(Y).Z + D\lambda_x(Z).\, Y - g_x(Y,Z).B_x]). \tag{2.14}$$

But, the curvature tensor $\overline{R}'_{\bar{x}}$ of the space M with respect to the linear connection $\overline{\Gamma}'_{\bar{x}}$ takes the form [2]:

$$R_x'(X;Y,Z) = D\Gamma_x'(\underline{Z};X,\underline{Y}) + \Gamma_x'(\Gamma_x'(X,\underline{Y}),\underline{Z}), \tag{2.15}$$

where R_x' is the model of $\overline{R}_{\bar{x}}'$ with respect to the chart $c = (U,\Phi,E)$ at the point $\bar{x} \in M$.

Differentiating both sides of equation (2.14) in the direction of a vector Z, we have:

$$D\Gamma_x'(\underline{Z};X,\underline{Y}) = D\Gamma_x(\underline{Z};X,\underline{Y}) + \frac{1}{2}[D^2\lambda_x(\underline{Z};X).\underline{Y} +$$
$$D^2\lambda_x(\underline{Z};\underline{Y}).X - Dg_x(\underline{Z};X,\underline{Y}).B_x - g_x(X,\underline{Y}).DB_x(\underline{Z})]. \tag{2.16}$$

Another time, from equation (2.14), we can get:

$$\Gamma_x'(\Gamma_x'(X,\underline{Y}),\underline{Z}) = \Gamma_x(\Gamma_x(X,\underline{Y}),\underline{Z}) - \frac{1}{2}D\lambda_x(\Gamma_x(X,\underline{Z})).\underline{Y} -$$
$$\frac{1}{4}D\lambda_x(X).D\lambda_x(\underline{Z}).\underline{Y} - \frac{1}{2}g_x(X,\underline{Y}).\Gamma_x(B_x,\underline{Z}) +$$
$$\frac{1}{4}g_x(X,\underline{Z}).D\lambda_x(B_x).\underline{Y} + \frac{1}{4}g_x(X,\underline{Y}).g_x(B_x,\underline{Z}).B_x - \frac{1}{2}g_x(\Gamma_x(X,\underline{Y}),\underline{Z})B_x. \tag{2.17}$$

Substituting from equations (2.16) and (2.17) into equation (2.15), we can obtain:

$$R_x'(X;Y,Z) = R_x(X;Y,Z) + [\frac{1}{2}\nabla(D\lambda_x(\underline{z};X)) -$$
$$\frac{1}{4}D\lambda_x(X).D\lambda_x(\underline{Z}) + \frac{1}{4}g_x(X,\underline{Z}).D\lambda_x(B_x)].Y -$$
$$\frac{1}{2}g_x(X,\underline{Y}).[\nabla B_x(\underline{Z}) - \frac{1}{2}g_x(B_x,\underline{Z}).B_x]. \tag{2.18}$$

Now, if denoting:

$$\omega_x(X) = \frac{1}{2}D\lambda_x(X), \tag{2.19}$$

$$P_x(X,Y) = \frac{1}{2}\nabla(D\lambda_x(X,Y)) - \frac{1}{4}D\lambda_x(X).D\lambda_x(Y) +$$
$$\frac{1}{8}g_x(X,Y).g_x(B_x,B_x), \tag{2.20}$$

we have:

$$P_x(X,Y) = \nabla\omega_x(X;Y) - \omega_x(X).\omega_x(Y) + \frac{1}{8}g_x(X,Y).g_x(B_x,B_x). \tag{2.21}$$

In this case, equation (2.18) takes the form:

$$R_x'(X;Y,Z) = R_x(X;Y,Z) + P_x(\underline{Z},X).\underline{Y} + g_x(X,\underline{Z}).[\frac{1}{2}\nabla B_x(\underline{Y}) - \frac{1}{4}g_x(B_x,\underline{Y}).B_x + \frac{1}{8}g_x(B_x,B_X).\underline{Y}]. \tag{2.22}$$

Using δ_x as a solution of the equation:

$$g_x(X,\delta_x(Y)) = P_x(X,Y), \tag{2.23}$$

then considering equations (2.20) and (2.23), we get:

$$g_x(X,\delta_x(Y)) = P_x(X,Y) = \frac{1}{2}\nabla(D\lambda_x(Y;X)) - \frac{1}{4}D\lambda_x(Y).D\lambda_x(X) + \frac{1}{8}g_x(Y,X).g_x(B_x,B_x). \tag{2.24}$$

From equation (2.13), we have:

$$\nabla D\lambda_x(Y;X) = \nabla g_x(Y;X,B_x) = g_x(X,\nabla B_x(Y)).$$

Applying this last equation into equation (2.24), yields:

$$g_x(Y,\delta_x(X)) = \frac{1}{2}g_x(Y,\nabla B_x(X)) - \frac{1}{4}D\lambda(X).g(Y,B_x) + \frac{1}{8}g_x(Y,X).g_x(B_x,B_x),$$

and since g is non-singular, we obtain:

$$\delta_x(X) = \frac{1}{2}\nabla B_x(X) - \frac{1}{4}g_x(X,B_x).B_x + \frac{1}{8}g_x(B_x,B_x).X. \tag{2.25}$$

Now, from equations (2.22) and (2.25), it is clear that:

$$R_x'(X;Y,Z) = R_x(X,Y,Z) + P_x(\underline{Z},X).Y + g_x(X,\underline{Z}).\delta_x(\underline{Y}). \tag{2.26}$$

Putting $R_x'(X;Y,Z) = 0,$ into equation (2.26), we have:

$$R_x(X;Y,Z) = P_x(\underline{Y},X).\underline{Z} + g_x(X,\underline{Y})\delta_x(\underline{Z}). \tag{2.27}$$

This means that, the equality (2.6) in the theorem, is satisfied.

By covariant differentiation of equation (2.19), locally with respect to $Y \in E$, we get:

$$\nabla\omega_x(Y;X) = \frac{1}{2}[D^2\lambda_x(Y;X) - D\lambda_x(\Gamma_x(Y,X)].$$

From this equation, we get:

$$\nabla\omega_x(\underline{Y};\underline{X}) = 0. \tag{2.28}$$

Using equations (2.21) and (2.28), we have:

$P_x(\underline{X},\underline{Y}) = 0,$ this means that, the tensor $\overline{P}_{\bar{x}}(\overline{X},\overline{Y})$ is symmetric. Furthermore, from equation (2.21) we get:

$$\nabla\omega_x(Y,Z) = P_x(Y,Z) + \omega_x(Y).\omega_x(Z) - \frac{1}{8}g_x(Y,Z).g_x(B_x,B_x). \tag{2.29}$$

Covariant differentiation of equation (2.29) locally with respect to $X \in E$ yields:

$$\nabla(\nabla\omega_x)(X;Y,Z)=\nabla P_x(X;Y,Z)+\nabla\omega_x(X;Y).\omega_x(Z)+$$
$$\omega_x(Y).\nabla\omega_x(X;Z)-\frac{1}{8}\nabla g_x(X;Y,Z).g_x(B_x,B_x)-$$
$$\frac{1}{8}g_x(Y,Z).\nabla g_x(X;B_x,B_x).$$

Using equation (2.29) into this last equation, we have:

$$\nabla(\nabla\omega_x)(X;Y,Z)=\nabla P_x(X;Y,Z)+\nabla\omega_x(X;Y).\omega_x(Z)+$$
$$\omega_x(Y).[P_x(X,Z)+\omega_x(X).\omega_x(Z)-\frac{1}{8}g_x(X,Z).g_x(B_x,B_x)]-$$
$$\frac{1}{4}g_x(Y,Z).g_x(B_x,\nabla B_x(X)).$$

Applying the alternation convention with respect to the vectors X,Y and using Ricci's identity [1] , we obtain the condition of complete integration of equation (2.29) as follows:

$$\nabla P_x(\underline{X};\underline{Y},Z)+\omega_x(\underline{Y}).P_x(\underline{X},\underline{Z})-\frac{1}{8}\omega_x(\underline{Y}).g_x(\underline{X},Z).g_x(B_x,B_x)-$$
$$\frac{1}{4}g_x(\underline{Y},Z).g_x(B_x,\nabla B_x(\underline{X}))+\omega_x(R_x(Z;Y,X))=0. \tag{2.30}$$

Now, using equations (2.25) and (2.27) into equation (2.30), we can get:

$$\nabla P_x(\underline{X};\underline{Y},Z)-\frac{1}{8}\omega_x(\underline{Y})g_x(\underline{X},Z)g_x(B_x,B_x)-$$
$$\frac{1}{4}g_x(\underline{Y},Z)g_x(B_x,\nabla B_x(\underline{X}))+\frac{1}{2}g_x(Z,\underline{Y})\omega_x(\nabla B_x(\underline{X})-\frac{1}{4}g_x(\underline{X},B_x)g_x(Z,\underline{Y})\omega_x(B_x)-$$
$$\frac{1}{8}g_x(Z,\underline{X})g_x(B_x,B_x)\omega_x(\underline{Y})=0. \tag{2.31}$$

Finally, applying equations (2.13) and (2.19) into equation (2.31), we obtain:-

$$\nabla P_x(\underline{X};\underline{Y},Z)=0, \tag{2.31}$$

for all $x=\Phi(\bar{x})\in\Phi(U)\subset E, X,Y,Z\in E.$ This means that, the equality (2.7) in the theorem, is satisfied.

Now, if $\dim M>3,$ we show that, the condition (2.7) follows directly from the condition (2.6) in the considered theorem:

In this case, we use Bianchi's identity [1] , which states that:

$$\nabla R_x(S;X;Y,Z)+\nabla R_x(Y;X;Z,S)+\nabla R_x(Z;X;S,Y)=0, \tag{2.33}$$

for all $x=\Phi(\bar{x})\in\Phi(U)\subset E, S,Y,Z,X\in E.$

Also, denoting:

$$g_x(S,R_x(X;Y,Z))=r_x(S,X,Y,Z), \tag{2.34}$$

and using the equations (2.23) and (2.27) into equation (2.34), we have:

$$r_x(X;Y;Z,W)=g_x(X,\underline{W})P_x(\underline{Z},Y)+g_x(Y,\underline{Z}).P_x(X,\underline{W}). \tag{2.35}$$

Applying identity (2.34) into equation (2.33), we get:

$$\nabla r_x(S;X;Y,Z,W)+\nabla r_x(Z;X;Y,W,S)+\nabla r_x(W;X;Y,S,Z)=0. \tag{2.36}$$

Covariant differentiation of equation (2.35) with respect to $S\in E$, we obtain:

$$\nabla r_x(S;X;Y,Z,W)=g_x(X,\underline{W}).\nabla P_x(S;\underline{Z},Y)+g_x(Y,\underline{Z}).\nabla P_x(S;X,\underline{W}).$$

Similarly, we get:

$$\nabla r_x(Z;X;Y,W,S) = g_x(X,\underline{S}).\nabla P_x(Z;\underline{W},Y) + g_x(Y,\underline{W}).\nabla P_x(Z;X,\underline{S}),$$

$$\nabla r_x(W;X;S,Z) = g_x(X,\underline{Z}).\ \nabla P_x(W;\underline{S},Y) + g_x(Y,\underline{S}).\ \nabla P_x(W;X,\underline{Z}).$$

Substituting these last three equations into equation (2.36), we have:

$$g_x(X,\underline{X}).\nabla P_x(S;\underline{Z},Y) + g_x(Y,\underline{Z}).\nabla P_x(S;X,\underline{W}) +$$
$$g_x(X,\underline{S}).\nabla P_x(Z;\underline{W},Y) + g_x(Y,\underline{W}).\nabla P_x(Z;X,\underline{S}) +$$
$$g_x(X,\underline{Z}).\nabla P_x(W;\underline{S},Y) + g_x(Y,\underline{S}).\nabla P_x(W;X,\underline{Z}) = 0 \qquad (2.37)$$

Applying lemma(2.2) into equation (2.37), we obtain:

$$g_x(Y,\underline{Z}).\nabla P_{x(}S;X,\underline{W}) + g_x(Y,\underline{W}).\nabla P_x(Z;X,\underline{S}) + g_x(Y,\underline{S}).\nabla P(W;X,\underline{Z}) = 0,$$

for all $x \in \Phi(U) \subset E, Y \in E.$

Taking into account , in the last equation , that g_x is non – singular yields:

$$Z.\nabla P_x(\underline{S};X\underline{W}) + W.\nabla P_{x_x}(\underline{Z};X,\underline{S}) + S.\nabla P_x(\underline{W};X,\underline{Z}) = 0.$$

Since Z is linearly independent of W and S , then we get :

$$\nabla P_x(\underline{S};\underline{W},X) = 0.$$

This means that , we have three arbitrary vectors $S,W,X \in E^3$, satisfy the equations: $g_x(X,W) = g_x(X,S) = 0,$ and satisfy, also the equation $\nabla P_x(\underline{S};\underline{W},X) = 0.$ Furthermore, since $P_x(W,X) \in L_2(E;IR),$ then $\nabla P_x(\underline{S};\underline{W},W) \in L_3(E;IR)$ is a trilinear, anti-symmetric form with respect to the vectors S and W . Hence, from this and by using lemma (2.3.5) [2], we deduce that, $\nabla P_x(\underline{S};\underline{W},X)$

can be represented as follows:

$$\nabla P_x(\underline{S};\underline{W},X) = \mu_x(\underline{S}).g_x(\underline{W},X), \qquad (2.38)$$

where $\mu_x \in L(E;IR)$ is a linear, continuous form. From equations (2.37) and (2.38), ewe can find:

$$\mu_x(S).g_x(X,W).g_x(Y,Z) - \mu_x(S)g_x(X,Z).g_x(Y,W) +$$
$$\mu_x(Z).g_x(X,Z)g_x(Y,W) - \mu_x(Z).g_x(X,W).g_x(Y,S) +$$
$$\mu_x(W).g_x(X,Z).g_x(Y,S) - \mu_x(W).g_x(X,S).g_x(Y,Z) = 0, \qquad (2.39)$$

for all $x \in \Phi(U) \subset E$ and for all $X,Y,S,W \in E.$

Remark (2.2):

Since $\dim M > 3,$ then for all $S,X,W,Z \in E,$ we can find $Y \in E$ such that $g_x(Y,S) = g_x(Y,Z) = 0$. Appling this remark, into equation (2.39), we get: $g_x(X,\ \mu_x(Z).g_x(W,Y).S - \mu_x(S).g_x(W,Y).Z) = 0,$ for all $x \in \Phi(U) \subset E,$ and for all $X \in E.$

Taking into account, that g is non-singular, we have: $\mu_x(Z).g_x(W,Y).S - \mu_x(S)g_x(W,Y).Z = 0,$ (2.40) for all $z \in \Phi(U) \subset E$ and for all $S,Z,W \in E.$

Assuming that the vector S is linearly independent of the vector Z , we obtain:

$\mu_x(Z).g_x(W,Y)=0,$ for all $x \in \Phi(U) \subset E$ and for all $Z,W \in E.$

Since W is arbitrary vector and the metric g is non-singular, we have: $\mu_x(Z)=0,$ for all $x \subset \Phi(U) \subset E, Z \in E.$ this means that $\mu \equiv 0.$ (2.41)

Hence, $\nabla P_x(\underline{Z};\underline{W},Y) \equiv 0,$ (2.42)

for all $x \in \Phi(U) \subset E$ and for all $Z,Y,W \in E,$ which is required.

Sufficiency:

For this aim, we assume that M is a Riemannian Banach manifold with a strong non-singular metric $\overline{g}.$ Also, we suppose that the curvature tensor $\overline{R}$ of the space M, satisfies the equality (2.27) with the condition (2.32) such that the tensor $\overline{P}_{\bar{x}}(\overline{X},\overline{Y})$ is symmetric. Then, we show that the space M is locally plane.

But, since the condition (2.32) is satisfied, then the equation (2.29) has a solution $\omega_x(Y)$. Also, the equation (2.19) will has a solution λ. In this case, we make the transformation $g_x{}'=e^{\lambda(x)}.g_x,$ and we get the Riemannian Banach manifold ($M,\overline{g}'$) with a curvature tensor $\overline{R}'\equiv 0.$ Hence the space $(M,\overline{g})$ is conformal to the locally plane space ($M,\overline{g}'$) and this completes the proof of the theorem.

Now, we introduce the following lemma:

Lemma (2.3): Let E be a vector space such that dim $E \geq 4,$ with a strong non-singular operator $g \in L_2(E;IR).$ If $X,Y \in E$ are arbitrary vectors such that $X \neq 0$ and X is perpendicular to Y with respect to the operator g, then there exists a vector $Z \in E$, such that Z is perpendicular to Y and the vectors X,Z are linearly independent.

Proof: We have the following two cases:

(1) If Y is a non-isotropic vector $(g(Y,Y) \neq 0)$ and X is perpendicular to Y, then X and Y are linearly independent vectors.

(2) If Y is an isotropic vector, then we, also have two cases:

(a) The vectors X and Y are linearly independent.

(b) The vectors X and Y are linearly dependent. These cases are considered as follows:

(1) In this case we have $g(Y,Y) \neq 0$ and since dim $E \geq 4$, then there exists a vector $S \in E$, which is linearly independent of the vectors X and Y. Furthermore, if S is not perpendicular to Y, then we can take a vector $Z \in E$ to be perpendicular to Y as follows:

$Z=\alpha X+g(S,Y).Y-g(Y,Y).S,$ where α is an arbitrary number. It is clear that the vectors Z and X are linearly independent and the lemma is valid in this case.

(2) (a) In the present case $g(Y,Y)=0$ and the vectors X,Y are linearly independent. Then, if we take $Z=Y,$ we get $g(Z,Y)=0$ such that the vectors X and Z are linearly independent and the lemma is true.

(2) (b) In this case $X \neq 0, X=mY, m \in IR$ is constant and $g(X,Y)=0.$ But the lemma is valid also. Since, if the lemma is not true, then there exists a vector $Z \in E$ such that Z is perpendicular to Y and the vectors X and Z are linearly dependent. And, in this case we have that $\dim <Y>^{\perp}=1,$ where $<Y>^{\perp}$ is the orthogonal complement [3] of the hypersurface $<Y>$. This means that dim $E=2$, which is a contradiction with the fact that

dim $E \geq 4.$ This completes the proof of the considered lemma.

3. Riemannian Banach manifolds of constant sectional curvature:

Let $(M,\bar{g})$ be a Riemannian Banach manifold of constant sectional curvature [2]. In this case, the curvature tensor $\bar{R}_{\bar{x}}(\bar{X}_3;\bar{X}_1,\bar{X}_2)$ on the Banach manifold M has the form [2]:

$$\bar{R}_{\bar{x}}(\bar{X}_3;\bar{X}_1,\bar{X}_2) = \lambda_{\bar{x}}[\bar{g}_{\bar{x}}(\bar{X}_3,\bar{X}_2).\bar{X}_1 - \bar{g}_{\bar{x}}(\bar{X}_1,\bar{X}_3).\bar{X}_2], \tag{3.1}$$

for all $\bar{x} \in M, \bar{X}_1, \bar{X}_2, \bar{X}_3 \in T_{\bar{x}}M,$ where $\lambda_{\bar{x}}$ is a real function of points of the space M and is called the Gaussian curvature of the manifold M. Now, we consider the following theorem:

Theorem (3.1): A Riemannian Banach manifold $(M,\bar{g})$ of constant sectional curvature, such that dim $M \geq 4$ is a locally plane space.

Proof: It is sufficient to prove this theorem locally with respect to a chart $c=(U,\Phi,E)$ at a point $\bar{x} \in M.$ We assume that the manifold M is of class $C^r(r \geq 3,\infty)$ with a strong non-singular metric $\bar{g}$ [2].

Now, the curvature tensor $\bar{R}_{\bar{x}}$ of the space M, with respect to a chart $c=(U,\Phi,E)$ at a point $\bar{x} \in M$ و takes the form:

$$R_{\infty}(X;Y,Z) = \lambda_x[g_x(Y,X).Z - g_x(Z,X).Y], \tag{3.2}$$

for all $x=\Phi(\bar{x}) \in \Phi(U) \subset E, X,Y,Z \in E.$ Where R_x and g_x are the models of the tensor $\bar{R}_{\bar{x}}$ and the metric $\bar{g}_{\bar{x}}$ with respect to the chart c, respectively. Hence, by using theorem (2.1) we will find a symmetric tensor $P_{\bar{x}}(X,Y)$ satisfies the following conditions:

$$\lambda_x[g_x(Y,X).Z - g_x(Z,X).Y] = P_x(Y,X).Z - P_x(Z,X).Y + g_x(X,Y).\delta_x(Z) - g_x(X,Z).\delta_x(Y), \tag{3.3}$$

$$\nabla P_x(\underline{X};\underline{Y},Z) = 0, \tag{3.4}$$

such that $P_x(Y,Z) = g_x(Z,\delta_x(Y)),$ (3.5)

for all $x \in \phi(U) \subset E, X,Y,Z \in E.$

Multiplying both sides of equation (3.3) by the arbitrary vector $S \in E$ and using the equality (3.5) , gives us:

$$\lambda_x[g_x(Y,X).g_x(S,Z) - g_x(Z,X).g_x(S,Y)] = P_x(Y,X).g_x(S,Z) - P_x(Z,X).g_x(S,Y) + g_x(X,Y).P_X(S,Z) - g_x(X,Z).P_x(Y,S). \tag{3.6}$$

Now, using lemma (2.1) we find that: for all $Y \neq 0$, $S \in E$ and S is perpendicular to Y, there exists a vector $Z \in E$ such that S is perpendicular to Z and the vectors Z,Y are linearly independent.

Hence, from equation (3.6) we get:

$$g_x(X,Y).P_x(S,Z) - g_x(X,Z).P_x(Y,S) = 0,$$

for all $x \in \phi(U) \subset E, X \in E.$

Since the metric g is non-singular, we obtain: $P_x(S,Z).Y - P_x(Y,S).Z = 0.$

Taking into account that the vectors Z and Y are linearly independent, we get: $P_x(Y,S)=0.$

Also, using lemma (2.3.3) [2] which states that: If for all a pair of vectors $(Y,S)\in E^2$ satisfies the condition $g_x(Y,S)=0,$ the following condition $P_x(Y,S)=0$ is also, satisfied, where $P_x\in L_2(E;IR).$ Then there exists a real number γ_x such that $P_x(X,Y)=\gamma_x.g_x(X,Y).$ (3.7)

Thus, from the relations (3.3) and (3.7) we have:

$$\lambda_x[g_x(Y,X).Z-g_x(Z,X).Y]=\gamma_x.g_x(Y,X).Z-\gamma_x g_x(Z,X).Y+g_x(X,Y).\delta_x(X)-g_x(X,Z).\delta_x(Y). \tag{3.8}$$

Also, from equations (3.5) and (3.7) we obtain : $\gamma_x.g_x(Y,Z)=g_x(Z,\delta_x(Y)),$ for all $x\in\Phi(U)\subset E, Y,Z\in E.$

But, since the metric g is non-singular, we get: $\delta_x(Y)=\gamma_x..Y.$

From this result and using equation (3.8), it is clear that:

$$\lambda_x[g_x(Y,X).Z-g_x(Z,X).Y]=2\gamma_x.g_x(Y,X).Z-2\gamma_x.g_x(Z,X).Y,$$

for all $x\in\Phi(U)\subset E, X,Y,Z\in E.$

Hence, by taking the vectors Z and Y are linearly independent we have:

$$\lambda_x.g_x(Y,X)=2\gamma_x.g_x(Y,X),$$

for all, $x\in\Phi(U)\subset E, X\in E.$

Since the metric g is non-singular and the vector X is arbitrary , we obtain: $g_x(X,Y)\neq 0.$ This means that $\lambda_x=2\gamma_{x.}$. From which and considering equation (3.7) yields:

$$P_x(X,Y)=\frac{\lambda_x}{2}g_x(X,Y). \tag{3.9}$$

Furthermore, the tensor $P_x(X,Y)$ satisfies the condition (3.4) which in the form: $\nabla P_x(\underline{S},\underline{X},Y)=0,$ for all $x\in\Phi(U)\subset E,S,X,Y\in E.$ Hence, the tensor $P_x(X,Y)$ satisfies all the required conditions and this completes the proof of the considered theorem.

4. The metric tensor of a Banach space of constant sectional curvature:

Let M be a Riemannian Banach manifold of constant sectional curvature λ_x [2] of class $C^r(r\geq 3),$ modeled on a Banach space E . Assume that the metric tensor $g_{\bar{x}}$ on the space M is strong non-singular [2].

Now, we consider the following theorem:

Theorem (4.1): If the metric tensor $g_{\bar{x}}$ on the manifold M , with respect to a chart $c=(U,\Phi,E)$ at the point $\bar{x}\in M$ has the form:

$$g_x(X,Y)=g^1(X,Y)/\Psi_x^{\ 2}, \tag{4.1}$$

for all $x\in\Phi(U)\subset E, X,Y\in E$. Where g^1 is a bilinear continuous symmetric strong non-singular, constant form, does not depend on the point $x\in\Phi(U)$ and is defined on the space E . Then the scalar function Ψ_x on the set $\Phi(U)$ will has the form:

$$\Psi_x = 1 + \frac{\lambda_x}{4}.g^1(x,x).$$

Proof: Differentiating the relation (4.1) with respect to the point $x \in \Phi(U) \subset E$ in the direction of the vector $Z \in E$, we get:

$$Dg_x(Z;X,Y) = \frac{-2g^1(X,Y).D\Psi_x(Z)}{\Psi^3{}_x},$$

similarly we have:

$$Dg_x(Y;X,Z) = \frac{-2g^1(X,Z).D\Psi_x(Y)}{\Psi^3{}_x},$$

$$Dg_x(X;Y,Z) = \frac{-2g^1(Y,Z).D\Psi_x(X)}{\Psi^3{}_x}.$$

Using the relations (1) and (4.1), we can obtain:

$$g^1(Z,\Gamma_x(X,Y)) = g^1(Z,\frac{-1}{\Psi_x}[D\Psi_x(X).Y + D\Psi_x(Y).X]) +$$

$$\frac{1}{\Psi_x}g^1(X,Y).D\Psi_x(Z). \tag{4.2}$$

Now, for all $x \in \Phi(U) \subset E$ we have that:

$D\Psi_x : X \in E \to D\Psi_x(X) \in IR$ is a linear continuous form [1] . And since the form g^1 is strong non-singular, then there exists a vector $B_x \in E$ such that:

$$D\Psi_x(X) = g^1(X,B_x), \tag{4.3}$$

for all $x \in \Phi(U) \subset E, X \in E.$

Hence, from equations (4.2), (4.3) and by taking into account that the form g^1 is non-singular, we can get:

$$\Gamma_x(X,Y) = \frac{1}{\Psi_x}[g^1(X,Y).B_x - D\Psi_x(X).Y - D\Psi_x(Y).X]. \tag{4.4}$$

Differentiating the relation (4.4) with respect to $x \in \Phi(U) \subset E$ in the direction of the vector $Z \in E$, we obtain:

$$D\Gamma_x(Z;X,Y) = \frac{1}{\Psi_x}[g^1(X,Y).DB_x(Z) - D^2\Psi_x(Z;X).Y -$$

$$D^2\Psi_x(Z;Y).X] - \frac{D\Psi_x(Z)}{\Psi^2{}_x}[g^1(X,Y).B_x - D\Psi_x(X).Y - D\Psi_x(Y).X]. \tag{4.5}$$

Also, from relation (4.4) we can have:

$$\Gamma_x(\Gamma_x(X,Y),Z = \frac{1}{\Psi_x^2}\{g^1(X,Y).g^1(B_x,Z).B_x - D\Psi_x(X).g^1(Y,Z).B_x -$$

$$D\Psi_x(Y),g^1(X,Z).B_x - g^1(X,Y).D\Psi_x(B_x).Z + 2D\Psi_x(X).D\Psi_x(Y).Z +$$

$$g^1(X,Y).D\Psi_x(Z).B_x + D\Psi_x(Z).D\Psi_x(X).Y +$$

$$D\Psi_x(Z).D\Psi_x(Y).X\}. \tag{4.6}$$

Now, from equations (4.5) and (4.6), we can get:

$$R_x(X;Y,Z) = D\Gamma_x(\underline{Z};X,\underline{Y}) - \Gamma_x(\Gamma_{x)_{\_}}(X,\underline{Y}),\underline{Z}) =$$

$$\frac{1}{\Psi_x}[g^1(X,\underline{Y}).DB_x(\underline{Z}) - D^2\Psi_x(\underline{Z};X).\underline{Y}] +$$

$$\frac{1}{\Psi_x^2} g^1(X,\underline{Z}).D\Psi_x(B_x).\underline{Y}, \tag{4.7}$$

where in this equation(4.7), $R_x(X;Y,Z)$ is the model of the curvature tensor $\overline{R}_{\bar{x}}(\overline{X};\overline{Y},\overline{Z})$ of the space M with respect to the chart c.

Since the space M has constant curvature [2], then by using equation (4.1) into equation (4.7) yields:

$$\frac{\lambda_x}{\Psi_x^2}[g^1(X,Y).Z - g^1(Z,X).Y] = \lambda_x[g_x(Y,X).Z - g_x(Z,X).Y] =$$

$$\frac{1}{\Psi_x} g^1(X,Y).DB_x(Z) - \frac{1}{\Psi_x} g^1(X,Z).DB_x(Y) -$$

$$\frac{1}{\Psi_x} D^2\Psi_x(Z;X).Y + \frac{1}{\Psi_x} D^2\Psi_x(Y;X).Z +$$

$$\frac{1}{\Psi_x^2} g^1(X,Z).D\Psi_x(B_x).Y - \frac{1}{\Psi_x^2} g^1(X,Y).D\Psi_x(B_x).Z, \tag{4.8}$$

for all $x \in \Phi(U) \subset E, X,Y,Z \in E.$

Now, assuming that dim $M \geq 4$ and using lemma (2.1) , we deuce that: for all arbitrary perpendcular vectors $X,Z \in E$ with respect to the form g^1 and $Z \neq 0$, there exists a vector $Y \in E$ such that X,Y are perpendicular with respect to g^1 and the vectors Y,Z are linearly independent. Hence, from this and using equation (4.8) we can obtain:

$$D^2\Psi_x(Z;X) = 0. \tag{4.9}$$

Also, considering lemma (2.3.3) [2],

then there exists a real number $\mu_x \in IR$ such that:

$$D^2\Psi_x(Z;X) = \mu_x.g^1(Z,X). \tag{4.10}$$

We will show that $\mu_{\bar{x}}$ is a scalar Quantity, does not depend on the point. Differentiating equation (4.3) in the direction of a vector $Y \in E$ and using (4.10), we get:

$$g^1(X, DB_x(Y)) = D^2\Psi_x(Y;X) = \mu_x.g^1(Y;X),$$

for all $x \in \Phi(U) \subset E, X,Y \in E.$

Since, the form g^1 is non-singular, we can obtain:

$$D^2B_x(X;Y) = D\mu_x(Y).X.$$

But $D^2B_x(X;Y) = D^2B_x(Y;X),$ from which, assuming that the vectors X,Y are linearly independent, we have:

$D\mu_x(X) = 0.$ This means that μ_x is a scalar, does not dependent on $x \in \Phi(U) \subset E.$ Hence, from (4.10) we deduce that: $D^2\Psi_x(X;Y) = \mu g^1(X,Y).$ (4.11)

Now, to find a solution for the differential equation (4.11) with respect to Ψ , we remark that:

$D^2\Psi_x = D(D\Psi_x) = \mu.g^1 \in L_2(E;IR) \cong L(E;L(E;IR)),$ is a constant function. Hence, if we put $D(D\Psi_x) = f,$ where $f \in L(E;L(E;IR)),$

then we get: $D\Psi_x = D\Psi(x) = f(x) + C,$

Where $C \in L(E;IR)$ is a constant function and $f(x) = \mu\, g^1(x,.) \in L(E;IR).$

Finally, we obtain:

$$\Psi_x = \Psi(x) = \frac{1}{2}(f(x))(x) + C(x) + C_0$$

$$= \frac{1}{2}\mu_x.g^1(x,x) + C(x) + C_0, \tag{4.12}$$

for all $x \in \Phi(U) \subset E$ such that $C_0 \in IR.$

Furthermore, all the solutions of equation (4.11) will be in the form (4.12). Since, if γ_x is another solution of the equation (4.8), then $\eta_X = \gamma_x - \Psi_x$ will be a solution of the equation: $D^2\eta_x \equiv 0.$

This means that $D\eta_x = h \in L(E;IR),$ is a constant function. And we get:

$\eta_x = \eta(x) = h(x) + h_0,$ for all $x \in \Phi(U) \subset E$ such that $h_0 \in IR$. From which, it is clear that:

$\gamma_x = \eta_x + \Psi_x = \frac{1}{2}\mu.g^1(x,x) + C(x) + C_0 + \ h(x) + h_0 = \frac{1}{2}\mu.g^1(x,x) + C_1(x) + C_2,$

where $C_1(x) = C(x) + h(x) \in L(E;IR),\ C_2 = C_0 + h_0 \in IR.$

This shows that, all the solutions of the differential equation (4.11) have the form (4.12).

Furthermore, since $C \in L(E;IR)$ is a covector and since the form g^1 is strong non-singular, then there exists a vector $A \in E$ such that:

$C(x) = g^1(A,X),$ for all $x \in \Phi(U) \subset E.$

From which and using (4.12), we obtain:

$$\Psi_x = \Psi(x) = \frac{1}{2}\mu.g^1(x,x) + g^1(A,X) + C_0. \tag{4.13}$$

Therefore, it is clear that:

$$D\Psi_x(Y) = \mu.g^1(x,Y) + g^1(A,Y). \tag{4.14}$$

Hence, we get:

$$D^2\Psi_x(Z;Y) = \mu.g^1(Z,Y), \tag{4.15}$$

for all $Z \in E.$

Also, by using equations (4.3) and (4.14) we deduce that: $g^1(B_x,Y) = D\Psi_x(Y) =$

$$= \mu.g^1(x,Y) + g^1(A,Y),$$

for all $x \in \Phi(U) \subset E, Y \in E.$

And we get: $g^1(Y, B_x - \mu.x - A) = 0,$ for all $x \in \Phi(U), Y \in E.$ Taking into account that g^1 is non-singular, we have $B_x = \lambda_x.x + A.$ (4.16)

Thus: $DB_x(Y) = \lambda_x.Y,$ (4.17)

for all $Y \in E.$

Similarly, considering equations (4.14) and (4.16), it is clear that:

$$D\Psi_x(B_x) = \mu^2.g^1(x,x) + g^1(A,A) + 2\mu\, g^1(x,A). \tag{4.18}$$

Now, applying equations (4.13), (4.15),(4.17), and (4.18) into equation (4.8) and then comparing the coefficients of the vector Z in both sides of the result, we can obtain:

$$\frac{\eta_x}{\Psi_x}.g^1(X,Y) = \frac{2\mu}{\Psi_x}.g^1(X,Y) - \frac{1}{\Psi_x^2}.g^1(X,Y).[\mu^2.g^1(x,x) + 2\mu.g^1(x,A) + g^1(A,A)],$$

for all $x \in \Phi(U) \subset E, X, Y \in E.$

From which, by considering equation (4.13) and using the non-singularity of g^1, we can have:

$$2\mu.C_0 - \lambda_x - g^1(A,A) = 0. \tag{4.19}$$

Now, to complete the proof of theorem (4.1), we must consider the following theorem:

Theorem (4.2): For a strong non-singular Riemannian metric $\overline{g}$ of a Banach Riemannian manifold M of constant sectional curvature λ_x, which represents, locally with respect to a chart $c = (U, \Phi, E)$ in the form: $g_x(X,Y) = (2) = \dfrac{g^1(X,Y)}{\Psi_x^2}$, where $\Psi_x = \Psi(x) = (14) = \dfrac{1}{2}\mu_x.g^1(x,x) + g^1(A,x) + C_0.$ such that the constants $\mu, C_0 \in IR$, and the vector $A \in E$ satisfy the condition (4.19), we can find another chart $c' = (U', \Phi', E')$ in which the metric $\overline{g}$ takes the form:

$$g'_{x'}(X',Y') = \frac{g^1(X',Y')}{[1 + \frac{\lambda_x}{4} g^1(x',x')]^2}, \tag{4.20}$$

which is a special case of the functions (4.1) and (4.13) when: $\mu = \dfrac{\lambda_x}{2}, C_0 = 1,$ and $A = 0.$

Proof: According to the values of the constants μ and C_0, the following cases are considered:

Case 1: If $\mu \neq 0,$ then equation (4.12), by taking into account the condition (4.19) takes the form:

$$\Psi_x = \frac{1}{2\mu}.g^1(\mu.x + A, \mu.x + A) + \frac{\lambda_x}{2\mu}. \tag{4.21}$$

Now, we consider the transformation:

$$x = F(x') = \frac{1}{\mu}\,[\frac{2x'}{g^1(x',x')} - A]. \tag{4.22}$$

Thus we have:

$$x' = F^{-1}(x) = \frac{2(\mu.x + A)}{g^1(\mu.x + A, \mu.x + A)}, \tag{4.23}$$

and this gives us a new chart $c' = (U', \Phi', E'),$ for which the metric $g_x(X,Y)$ takes the form:

$$g_x(X,Y) = \frac{g^1(X,Y)}{(\Psi(x))^2} = \frac{g^1(DF_{x^1}(X'), DF_{x^1}(Y'))}{[\Psi(F(x'))]^2}, \tag{4.24}$$

where $DF_{x'}(X') = \dfrac{2}{\mu}[\dfrac{X'}{g^1(x',x')} - \dfrac{x'.2g^1(x',x')}{g^1(x',x'))^2}],$ (4.25)

$$DF_{x'}(Y') = \frac{2}{\mu}[\frac{Y'}{g^1(x',x')} - \frac{x'.2g^1(x',Y')}{g^1(x',x'))^2}].$$

and similarly (4.26)

Hence, from equations (4.21), (4.24), (4.25) and (4.26) we can get:

$$g_x(X,Y) = \frac{g^1(X',Y')}{([1+\frac{\lambda_x}{4}g^1(x',x')]^2} = g'_{x'}(X',Y'),$$

which is required.

Case 2: If $\mu = 0$ and $C_0 \neq 0,$, then equation (4.12) takes the form:

$$\Psi(x) = g^1(A,x) + C_0. \tag{4.27}$$

Also, the condition (4.22) becomes:

$$g^1(A,A) = -\lambda_x. \tag{4.28}$$

Then, we consider the transformation:

$$x = F(x^1) = \frac{2x'}{g^1(x',x')}, \tag{4.29}$$

and

$$x' = F^{-1}(x) = \frac{2x'}{g^1(x,x)}. \tag{4.30}$$

With respect to this transformation, the metric g_x has the form:

$$g_x(X,Y) = \frac{g^1(X,Y)}{[g^1(A,x)+C_0]^2} = \frac{g^1(DF_{x'}(X'),DF_{x'}(Y'))}{[\Psi(F(x'))]^2} =$$

$$= \frac{g^1(X',Y')}{[{C_0}/{2}.g^1(x',x') + g^1(A,x')]^2},$$

which is the first case with $\mu' = C_0 \neq 0, A' = A$ and $C_0' = 0.$

Case 3: If $\mu = 0$ and $C_0 = 0,$ then we obtain:

$$\Psi(x) = g^1(A,x), \tag{4.31}$$

and $g^1(A,A) = -\lambda_x.$

Hence $A \neq 0$, and since the form g^1 is a strong non-singular, then there exists a vector $S \in E$ such that:

$$g^1(A,S) = S_0 \neq 0. \tag{4.32}$$

Thus, by considering the transformation $F(x') = x = x' + S, x' = x - S = F^{-1}(x),$ then the metric g will be in the form:

$$g_x(X,Y) = \frac{g^1(X,Y)}{[g^1(A,X)]^2}.$$

This means that:

$$g_x(X,Y) = \frac{g^1(DF_x(x'),DF_x(Y'))}{[g^1(A,F(x'))]^2} =$$

$$\frac{g^1(X',Y')}{[g^1(A,x')+S_0]^2}, \tag{4.33}$$

which is the second case with $A' = A$ and $C'_0 = S_0$. This completes the proof of theorem (4.1). Hence, in the case of a Riemannian Banach manifold of constant Gaussian curvature, and at any point $\bar{x} \in M$, there exists a chart $c' = (U', \Phi', E')$, such that the metric tensor of this space takes the canonical form (4.20) with respect to this chart. Which is a generalization of this result in the finite-dimensional Riemannian geometry.

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