On the Fundamental Theorem in Arithmetic Progression of Primes

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Abstract: Using Jiang function we prove the fundamental theorem in arithmetic progression of primes. The primes contain only $k < P_{g+1}$ long arithmetic progressions, but the primes have no $k > P_{g+1}$ long arithmetic progressions theorem.

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Theorem. The fundamental theorem in arithmetic progression of primes.

We define the arithmetic progression of primes [1-3].

$$P_{i+1} = P_1 + \omega_g i, i = 0,1,2,\cdots,k-1, \tag{1}$$

where $\omega_g = \prod_{2 \le P \le P_g} P$ is called a common difference, P_g is called g -th prime.

We have Jiang function [1-3]

$$J_2(\omega) = \prod_{3 \le P}(P - 1 - X(P)), \tag{2}$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k-1}(q + \omega_g i) \equiv 0 (\text{mod } P), \tag{3}$$

where $q = 1,2,\cdots,P-1$.

If $P|\omega_g$, then $X(P) = 0$; $X(P) = k-1$ otherwise. From (3) we have

$$J_2(\omega) = \prod_{3 \le P \le P_g}(P-1) \prod_{P_{g+1} \le P}(P-k). \tag{4}$$

If $k = P_{g+1}$ then $J_2(P_{g+1}) = 0$, $J_2(\omega) = 0$, there exist finite primes P_1 such that $P_2,\cdots,P_k$ are primes. If $k < P_{g+1}$ then $J_2(\omega) \ne 0$, there exist infinitely many primes P_1 such that $P_2,\cdots,P_k$ are primes. The primes contain only $k < P_{g+1}$ long arithmetic progressions, but the primes have no $k > P_{g+1}$ long arithmetic progressions. We have the best asymptotic formula [1-3]

$$\pi_k(N,2) = \left|\{P_1 + \omega_g i = \text{prime}, 0 \le i \le k-1, P_1 \le N\}\right|$$

$$= \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)} \frac{N}{\log^k N}(1 + o(1)), \tag{5}$$

where $\omega = \prod_{2 \le P} P, \phi(\omega) = \prod_{2 \le P}(P-1)$, ω is called primorial, $\phi(\omega)$ Euler function.

Suppose $k = P_{g+1} - 1$. From (1) we have

$$P_{i+1} = P_1 + \omega_g i, i = 0,1,2,\cdots,P_{g+1}-2. \tag{6}$$

From (4) we have [1-2]

$$J_2(\omega)=\prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-P_{g+1}+1)\to\infty \text{ as } \omega\to\infty \tag{7}$$

We prove that there exist infinitely many primes P_1 such that $P_2,\cdots,P_{P_{g+1}-1}$ are primes for all P_{g+1}.
From (5) we have

$$\pi_{P_{g+1}-1}(N,2)=$$

$$\prod_{2\le P\le P_g}\left(\frac{P}{P-1}\right)^{P_{g+1}-2}\prod_{P_{g+1}\le P}=\frac{P^{P_{g+1}-2}(P-P_{g+1}+1)}{(P-1)^{P_{g+1}-1}}\frac{N}{(\log N)^{P_{g+1}-1}}(1+o(1)). \tag{8}$$

From (8) we are able to find the smallest solutions $\pi_{P_{g+1}-1}(N,2)>1$ for large P_{g+1}.

Theorem is foundation for arithmetic progression of primes

Example 1. Suppose $P_1=2,\ \omega_1=2,\ P_2=3$. From (6) we have the twin primes theorem

$$P_2=P_1+2. \tag{9}$$

From (7) we have

$$J_2(\omega)=\prod_{3\le P}(P-2)\to\infty \text{ as } \omega\to\infty, \tag{10}$$

We prove that there exist infinitely many primes P_1 such that P_2 are primes. From (8) we have the best asymptotic formula

$$\pi_2(N,2)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N}{\log^2 N}(1+o(1)). \tag{11}$$

Twin prime theorem is the first theorem in arithmetic progression of primes.

Example 2. Suppose $P_2=3,\ \omega_2=6,\ P_3=5$. From (6) we have

$$P_{i+1}=P_1+6i,\ i=0,1,2,3. \tag{12}$$

From (7) we have

$$J_2(\omega)=2\prod_{5\le P}(P-4)\to\infty \text{ as } \omega\to\infty, \tag{13}$$

We prove that there exist infinitely many primes P_1 such that P_2, P_3 and P_4 are primes. From (8) we have the best asymptotic formula

$$\pi_4(N,2)=27\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N}{\log^4 N}(1+o(1)). \tag{14}$$

Example 3. Suppose $P_9=23,\ \omega_9=223092870,\ P_{10}=29$. From (6) we have

$$P_{i+1}=P_1+223092870i,\ i=0,1,2,\cdots,27. \tag{15}$$

From (7) we have

$$J_2(\omega)=36495360\prod_{29\le P}(P-28)\to\infty \text{ as } \omega\to\infty, \tag{16}$$

We prove that there exist infinitely many primes P_1 such that $P_2,\cdots,P_{28}$ are primes. From (8) we have the best asymptotic formula

$$\pi_{28}(N,2)=\prod_{2\le P\le 23}\left(\frac{P}{P-1}\right)^{27}\prod_{29\le P}\frac{P^{27}(P-28)}{(P-1)^{28}}\frac{N}{\log^{28}N}(1+o(1)). \quad (17)$$

From (17) we are able to find the smallest solutions $\pi_{28}(N_0,2)>1$.

On May 17, 2008, Wroblewski and Raanan Chermoni found the first known case of 25 primes: $6171054912832631+366384\times\omega_{23}\times n$, for $n=0$ to 24.

Theorem can help in finding for 26, 27, 28, …, primes in arithmetic progressions of primes.

Corollary 1. Arithmetic progression with two prime variables

Suppose $\omega_g=d$. From (1) we have

$$P_1, P_2=P_1+d, P_3=P_1+2d,\cdots,P_k=P_1+(k-1)d,(P_1,d)=1. \quad (18)$$

From (18) we obtain the arithmetic progression with two prime variables: P_1 and P_2,

$$P_3=2P_2-P_1,\quad P_j=(j-1)P_2-(j-2)P_1,\quad 3\le j\le k<P_{g+1}. \quad (19)$$

We have Jiang function [3]

$$J_3(\omega)=\prod_{3\le P}[(P-1)^2-X(P)], \quad (20)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=3}^{k}[(j-1)q_2-(j-2)q_1]\equiv 0(\bmod P), \quad (21)$$

where $q_1=1,2,\cdots,P-1;\ q_2=1,2,\cdots,P-1$.

From (21) we have

$$J_3(\omega)=\prod_{3\le P\le k}(P-1)\prod_{k<P}(P-1)(P-k+1)\to\infty \text{ as } \omega\to\infty. \quad (22)$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3,\cdots,P_k$ are primes for $3\le k<P_{g+1}$

we have the best asymptotic formula

$$\pi_{k-1}(N,3)=\left|\{(j-1)P_2-(j-2)P_1=\text{prime},3\le j\le k,P_1,P_2\le N\}\right|$$

$$=\frac{J_3(\omega)\omega^{k-2}}{\phi^k(\omega)}\frac{N^2}{\log^k N}(1+o(1)), \quad (23)$$

From (23) we have the best asymptotic formula

$$\pi_{k-1}(N,3)=\prod_{2\le P\le k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k<P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \quad (24)$$

From (24) we are able to find the smallest solution $\pi_{k-1}(N_0,3)>1$ for large $k<P_{g+1}$.

Example 4. Suppose $k=3$ and $P_{g+1}>3$. From (19) we have

$$P_3=2P_2-P_1. \quad (25)$$

From (22) we have

$$J_3(\omega)=\prod_{3\le P}(P-1)(P-2)\to\infty \text{ as } \omega\to\infty, \quad (26)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 are primes. From (24) we have the best asymptotic formula

$$\pi_2(N,3) = 2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N^2}{\log^3 N}(1+o(1)) = 1.32032\frac{N^2}{\log^3 N}(1+o(1)). \quad (27)$$

Example 5. Suppose $k=4$ and $P_{g+1}>4$. From (19) we have

$$P_3 = 2P_2 - P_1, \qquad P_4 = 3P_2 - 2P_1. \quad (28)$$

From (22) we have

$$J_3(\omega) = 2\prod_{5\le P}(P-1)(P-3) \to \infty \text{ as } \omega\to\infty, \quad (29)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 and P_4 are primes. From (24) we have the best asymptotic formula

$$\pi_3(N,3) = \frac{9}{2}\prod_{5\le P}\frac{P^2(P-3)}{(P-1)^3}\frac{N^2}{\log^4 N}(1+o(1)). \quad (30)$$

Example 6. Suppose $k=5$ and $P_{g+1}>5$. From (19) we have

$$P_3 = 2P_2 - P_1, \quad P_4 = 3P_2 - 2P_1, \quad P_5 = 4P_2 - 3P_1. \quad (31)$$

From (22) we have

$$J_3(\omega) = 2\prod_{5\le P}(P-1)(P-4) \to \infty \text{ as } \omega\to\infty, \quad (32)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3, P_4 and P_5 are primes. From (24) we have the best asymptotic formula

$$\pi_4(N,3) = \frac{27}{2}\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N^2}{\log^5 N}(1+o(1)). \quad (33)$$

Corollary 2. Arithmetic progression with three prime variables

From (18) we obtain the arithmetic progression with three prime variables: P_1, P_2 and P_3

$$P_4 = P_3 + P_2 - P_1, \quad P_j = P_3 + (j-3)P_2 - (j-3)P_1, \quad 4\le j\le k< P_{g+1} \quad (34)$$

We have Jiang function

$$J_4(\omega) = \prod_{3\le P}((P-1)^3 - X(P)), \quad (35)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=4}^{k}(q_3 + (j-3)q_2 - (j-3)q_1) \equiv 0 \pmod P, \quad (36)$$

where $q_i = 1,2,\cdots,P-1,\ i=1,2,3$.

Example 7. Suppose $k=4$ and $P_{g+1}>4$. From (34) we have

$$P_4 = P_3 + P_2 - P_1. \quad (37)$$

From (35) and (36) we have

$$J_4(\omega) = \prod_{3\le P}(P-1)(P^2-3P+3) \to \infty \text{ as } \omega\to\infty, \quad (38)$$

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that P_4 are primes. We have the best asymptotic formula

$$\pi_2(N,4)=2\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\frac{N^3}{\log^4 N}(1+o(1)). \qquad (39)$$

For $k\ge 5$ from (35) and (36) We have Jang function

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2$$
$$\times\prod_{(k-1)\le P}(P-1)[(P-1)^2-(P-2)(k-3)]\to\infty$$

as $\omega\to\infty$. (40)

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that $P_4,\cdots,P_k$ are primes for $5\le k<P_{g+1}$.

we have the best asymptotic formula

$$\pi_{k-2}(N,4)=\left|\{P_3+(j-3)P_2-(j-3)P_1=\text{prime},4\le j\le k,P_1,P_2,P_3\le N\}\right|$$
$$=\frac{J_4(\omega)\omega^{k-3}}{\phi^k(\omega)}\frac{N^3}{\log^k N}(1+o(1)). \qquad (41)$$

From (41) we have

$$\pi_{k-2}(N,4)$$
$$=\prod_{2\le P<(k-1)}\frac{P^{k-3}}{(P-1)^{k-2}}\prod_{(k-1)\le P}\frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}}\frac{N^3}{\log^k N}(1+o(1)). \qquad (42)$$

From (42) we are able to find the smallest solution $\pi_{k-2}(N_0,4)>1$ for large $k<P_{g+1}$.

Corollary 3. Arithmetic progression with four prime variables

From (18) we obtain the arithmetic progression with four prime variables: P_1,P_2,P_3 and P_4

$P_5=P_4+2P_3-3P_2+P_1$, $\quad P_j=P_4+(j-3)P_3-(j-2)P_2+P_1$,

$5\le j\le k<P_{g+1}$ (43)

We have Jiang function

$$J_5(\omega)=\prod_{3\le P}\left[(P-1)^4-X(P)\right], \qquad (44)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=5}^{k}[q_4+(j-3)q_3-(j-2)q_2+q_1]\equiv 0 \pmod P, \qquad (45)$$

where

$$q_i=1,\cdots,P-1,i=1,2,3,4$$

Example 8. Suppose $k=5$ and $P_{g+1}>5$. From (43) we have

$$P_5=P_4+2P_3-3P_2+P_1. \qquad (46)$$

From (44) and (45) we have

$$J_5(\omega)=12\prod_{5\le P}(P-1)(P^3-4P^2+6P-4)\to\infty$$ as $\omega\to\infty$. (47)

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that P_5 are primes.
We have the best asymptotic formula

$$\pi_2(N,5)=\frac{J_5(\omega)\omega}{\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+o(1)). \quad (48)$$

Example 9. Suppose $k=6$ and $P_{g+1}>6$. From (43) we have

$$P_5=P_4+2P_3-3P_2+P_1, \qquad P_6=P_4+3P_3-4P_2+P_1. \quad (49)$$

From (44) and (45) we have

$$J_5(\omega)=10\prod_{5\le P}(P-1)(P^3-5P^2+10P-9)\to\infty$$ as $\omega\to\infty$. (50)

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that P_5 and P_6 are primes.
We have the best asymptotic formula

$$\pi_3(N,5)=\frac{J_5(\omega)\omega^2}{\phi^6(\omega)}\frac{N^4}{\log^6 N}(1+o(1)). \quad (50)$$

For $k\ge 7$ from (44) and (45) we have Jiang function

$$J_5(\omega)=6\prod_{5\le P\le(k-4)}(P-1)(P^2-3P+3)$$
$$\times\prod_{(k-4)<P}\left\{(P-1)^4-(P-1)^2\left[(P-3)(k-4)+1\right]-(P-1)(2k-9)\right\}\to\infty$$

as $\omega\to\infty$ (51)

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that $P_5,\cdots,P_k$ are primes.
We have best asymptotic formula

$$\pi_{k-3}(N,5)=\left|\left\{P_4+(j-3)P_3-(j-2)P_2+P_1=\text{prime}, 5\le j\le k, P_1,\cdots,P_4\le N\right\}\right|$$
$$=\frac{J_5(\omega)\omega^{h-4}}{\phi^k(\omega)}\frac{N^4}{\log^k N}(1+o(1)). \quad (52)$$

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References

1. Chun-Xuan Jiang, On the prime number theorem in additive prime number theory, Preprint, 1995.
2. Chun-Xuan Jiang, The simplest proofs of both arbitrarily long arithmetic progressions of primes, preprint, 2006.
3. Chun-Xuan, Jiang, Foundations of Santiili's isonumber theory with applications to new cryptograms, Fermat's theorem and Goldbach's conjecture, Inter. Acad. Press, 68-74, 2002, MR 2004c: 11001, http://www.i-b-r.org/docs/jiang/pdf.

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