Uniformly Convex Spaces

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ABSTARACT: This paper depicts the class OF Banach Spaces with normal structure and that they are generally referred to as uniformly convex spaces. A review of properties of a uniformly convex spaces is also examined with a view to show that all non-expansive mapping have a fixed point on this space. With the definition of uniformly convex spaces in mind, we also proved that some spaces are uniformly spaces. [Researcher. 2009;1(1):74-85]. (ISSN: 1553-9865).

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INTRODUCTION

The importance of Uniformly convex spaces in Applied Mathematics and Functional Analysis, it has developed into area of independent research, where several areas of Mathematics such as Homology theory, Degree theory and Differential Geometry have come to play a very significant role. [1,3,4]
Classes of Banach spaces with normal structure are those generally refer to as Uniformly convex spaces. In this paper, we review properties of the space and show that all non-expansive maps have a fixed-point on this space. [2]

Let x be a Banach Space. A Branch space x is said to be Uniformly convex if for $\varepsilon > 0$ there exist a $\partial = (\varepsilon) > 0$ such that if x, y £ x with //x//=1, //y//=1 and //x-y//$\geq \varepsilon$, then $// \frac{1}{2}(x+y)// \leq 1-\partial$.

THEOREM (1.0)

Let x = Lp (μ) denote the space of measurable function f such that //f// are integrable, endowed with the norm.

$$// f // = (\partial x / f / \partial \mu)^{1/P}$$

Then for 1<P<+∞, the space $LP(\mu)$ is uniformly convex for the proof of the above theorem, we need the following basic lemma.

Lemma (1.0)

Let X=Lp, then for p q>0, such $\frac{1}{p}+\frac{1}{q}=1$ and for each pair f, g £ x, the following inequalities hold.

(i) For 1<p≤ 2

$$// \tfrac{1}{2}(f+g)// q + // \tfrac{1}{2}(f+g)// q \leq // 2^{-1}(// f // p + // g // q)$$

And

(ii) For $2 \leq p < \infty$

$$//(f+g // p + //(f-g // p \leq // 2^{-p}(// f // pp + // g // q)$$

We now apply lemma (1.0) to prove theorem (1.0)

Proof of theorem (1.0)

Choose $f_1 g\varepsilon X = Lp,$ such that $// f // \leq 1, // g // \leq 1$ and for any $\varepsilon > 0$, we have $// f - g // \geq \varepsilon$. Two cases arise:

Case1: $1 < p \leq 2$

In this case (1.0) yield

$$// \tfrac{1}{2}(f+g) // q + // \tfrac{1}{2}(f-g) // q \leq 2 - (q-1)(// f // p + // g //)^{q-1} //$$
$$\leq 2 - (q-1)(2)(q-1) = 1$$

Thus, $// \tfrac{1}{2}(f+g) // q \leq 1 - / \dfrac{f-g}{g} / q \leq 1 - \dfrac{(\varepsilon)^q}{2}$

Or $// \tfrac{1}{2}(f+g) // \leq 1 - \dfrac{(\varepsilon)^q}{2} //^{1/p > 0}$

So that by choosing $\delta = 1 - // \dfrac{(\varepsilon)}{2} // \; 1/q > 0$

We obtain $// \tfrac{1}{2}(f+g) // \leq 1 - \delta$ and so $x = Lp(kp \leq 2)$ is uniformly convex.

Case 2: $2 \leq p < \infty$

As in Case 1, we use (ii) of Lemma (3.1.1) to show that $x = Lp(kp < \infty)$ is uniformly convex, completing the proof of the theorem.

Since Lemma (3.1.1) is also valid for 1p, $1 \leq p < \infty$, the following theorem is also true.

Theorem (2.0)

For $1 \leq p < \infty$, the space 1p of all infinite (real or complex) sequence, (x_1,x_2,x_3,……) such that $\sum_{i=1}^{\infty} / x_i / p < +\infty$ is uniformly convex. As a special case of theorem (1.0), we have the following.

Corollary (1.0)

Every Hibert space H is Uniformly convex. Although theorem (1.0) and (3.0) provide large classes of space which are Uniformly convex, a few well known spaces are known not to be Uniformly convex.

1. The Space λ is not Uniformly convex

To see this $\varepsilon = 1$ and choose $\pi(1,0,0,0,.........), // Y // \lambda_1 // \pi - Y // \lambda_1 = 2 > \varepsilon..$ However, $// \tfrac{1}{2}(\pi - Y // \leq 1)$ and there is no $\varepsilon = 0$ such that $// \tfrac{1}{2}(X - Y // \leq 1 - \delta$, showing that λ_1 is not uniformly convex.

2. The space λ_∞ is not uniformly convex

Consider $U = (1,1,0,0,0,............,)$ and $V = (1,1,0,0,0,..........)$, Both U and V $\varepsilon\lambda_\infty$. Take $\varepsilon = 1$, then $// U // \infty = 1, // V // \infty$ and $//U - V//\infty = 2 > \varepsilon$. However, $// \tfrac{1}{2}(U - V) // = 1$ and so λ_∞ is not uniformly convex.

3. Consider c(a,b) the space of real-valued continuous function on the compact interval (a,b) with Sup norm.

 Then C (a,b) is not uniformly convex.

To see this, choose two function f(t), g(t) defined as follows:

F(t) = 1 for all $t\varepsilon(a,b)$

And

$g(t)=\dfrac{b-t}{b-a}$ for each $t\varepsilon(a,b)$

Take $\varepsilon=\dfrac{1}{2}$ clearly f(t), g(t) $\varepsilon\,\mathrm{c}(a,b)$. $//f//-//g//=1$ and $//f-g//\Rightarrow\varepsilon$.

Also, $//\tfrac{1}{2}(f+g)//=1$ and so c(a, b) is Not uniformly convex.

The following propositions are the consequences of the definition of uniform convexity.

Proposition (1.0)

Suppose x is Uniformly convex Banach space, then for any $\partial>0,\varepsilon>0$ arbitrary Vectors x,y£x with //x//$\leq\partial$, //y//$\geq$, there exists a $\delta>0$ such that $//\tfrac{1}{2}(x+y)//\leq\left\{1-\delta\dfrac{(\varepsilon)}{\partial}\right\}\partial$

Proof

Let $\varepsilon>0$ be given and let $z_1=x/\partial, z_2=y/\partial$ and suppose we set $\varepsilon=\varepsilon/d$.

$$//z_{1/2}//\leq 1 \text{ and } //z_1-z_2//=\tfrac{1}{0}//x-y//\geq^{\varepsilon}/d=\varepsilon$$

Now, by uniform convexity, we have

$$=//\tfrac{1}{2}z_1+z_2//\leq 1-\delta(\varepsilon)$$

That is $//\tfrac{1}{2}\partial(x+y)//\leq 1\delta\dfrac{(\varepsilon)}{4}$

Which implies, $//\tfrac{1}{2}(x+y)//\leq\left\{1-\delta\dfrac{(\varepsilon)}{\partial}\right\}\partial$

Proposition (2.0)

Let x be a uniformly convex Banach space. Then for any $\varepsilon>0,\varepsilon>0$ and $\partial\varepsilon(0,1)$ if x, yεX such that $//x//\leq\partial, //x-y//>\varepsilon$, then exist a;

$\delta=\partial(\varepsilon/\partial)>0$ such that

$$//\alpha x+(1-\alpha)y//-2^{(\varepsilon)}1/\partial\ \min\alpha,1-\alpha)//d$$

Proof

Without loss of generality, we may take $\varepsilon(0,\tfrac{1}{2})$

Now, $//\alpha x+(1-\alpha)y//=//\alpha(+y)+(1-2\alpha)//y//$

$$\leq 2\alpha//\tfrac{1}{2}(x+y)//+(1-2\alpha)//y//\text{..............................(1.0)}$$

But by proposition (1.0), we have that there exists a $\delta>0$, such that $//\tfrac{1}{2}(x+y)//\leq//-\delta(\varepsilon)//\partial$

Substitute this into (1.0), to have

Since //y// $\leq \partial$ $\quad = \left\{\partial - 2\alpha\delta \frac{(\varepsilon)}{\partial}\partial\right\}$

$$= \left(1 - 2\alpha\partial \frac{(\varepsilon)}{\partial}\partial\right)$$

Put by the choice of $\alpha\varepsilon\left(0, \frac{1}{2}\right)$, have $\alpha \geq \min(\alpha, 1 - \alpha)$

Thus, we have,

$$// \alpha x + (1 - \alpha)// \leq //1 - 2\delta \frac{(\varepsilon)}{\partial}\min(\alpha, 1 - \alpha)// \partial$$

We now discuss a characteristic of some Banach space, which is related to uniform convexity.

2.0 STRICTLY CONVEX BANACH SPACES

Definition (1.0)

A Banach space X is said to be strictly convex (or strictly rotund if for any pair of vecors x, y £ x, the equation //x + y//=//x+//y//, implies that there exists a $\lambda \geq 0$ such that $x = \lambda x\ (\text{or } y = \lambda x)$.

The following Lemma on uniform convexity will be useful in the sequence.

<u>Lemma</u> (1.0)

Let X be a uniformly convex Banach space.

If $\lambda + 0, // x - \lambda y// \geq 0$, then $// \frac{1}{2}(x + \lambda y)// \leq // x //$

Proof

Suppose $0 < // x // < // y //$. (The proof for the case $0 < // x // < // y //$ is similar).

Take $\lambda = \frac{// x //}{// y //}$ and set $a = x, b = \lambda y$. The $// a - b // = // x - \lambda y // > 0$,

Let $\varepsilon > 0$ such that $//x - \lambda y // \geq \varepsilon$

Observe that,

$$// a // = //x//, // b // = \lambda // y // = \frac{// x //// y //}{// y //} = // x //$$

So by proposition (3.1.1), there exist a $\delta = \frac{\delta(\varepsilon) > 0}{// x //}$

Such that $// \frac{1}{2}(a + b)// < 1 - \frac{\partial(\varepsilon)}{// x //}// x // < // x //,$

That is $// \frac{1}{2}(x + y)// < // x //.$

Completing the proof of the lemma, we now prove the following theorem.

<u>Theorem</u> (1.2)

Every uniformly convex space is strictly convex

Proof

Suppose x is uniformly convex Let x, $y\varepsilon X$ be non-zero vectors such that //x+y//=//x//+//y//

We need to show that there exist $a\lambda > 0$ such that $x = \lambda y$. We consider two possible cases.

Case I: // x //=// y //

Case II: $0 < // y// < // x //$, (The other case $0 < // x // < / y //$ is treated)

Similarly,

Proof of Case I

If x=, then (II) holds with $\lambda = 1$, so, suppose $x \neq y; \text{then as } \frac{// x //}{// y //} = 1$ and // x+y //<2 //x // =+ // y // (Since // x // y //).
That is, // x + y // <1 // x // + //y //. Contradicting (I).
Thus, x ≠ y is not possible and this proves case I.

Proof of Case 2

Suppose // x=y // = // x // + // y // and that x ≠ λ y. Since x is uniform convex, lemma (2.0) yields.

$$// \tfrac{1}{2}(x + \lambda y)// < // x // \qquad ....(1.1)$$

For $0 < // y // < // x//, \text{let } \lambda = \frac{// x //}{// y //} // x + y // = // x// + //y//$

And x ≠ λ y

We have $\lambda(// x // + // y //) = \lambda //x + y// = //\lambda x + \lambda y - x + x //$

$$\leq // x + \lambda y // + (\lambda + 1)// x // ....(1.2)$$

That is, $\lambda(// x // + // y //) \leq //x // + \lambda // y// + \lambda // x// - // x //$

$$= \lambda(// x // + // y //)................(1.3)$$

The inequalities (b) and (c) gives

$$\lambda(// x // + // y //) = // x + \lambda y // + (\lambda - 1)// x //$$

From which we obtain (since $\lambda = \frac{// x //}{// y //} \tfrac{1}{2}(x + \lambda y) = // x //$ Contracting (1.1). This completes the proof of the theorem.

Theorem (3.2.2) gives a large class of strict convex Banach spaces. However, it can be shown easily that $\lambda_1, L_1, \lambda_\infty$ and c(a,b) are NOT strictly convex. For example, to see that λ_∞ is not strictly convex.

3.0 THE MODLUS OF CONVEXITY

<u>Definition</u> **(2.0)**

Let x be a Banach space, the modulus of convexity of X is the function $\delta x : (0,2) \rightarrow (0,1)$ defined by $\delta x(\varepsilon) = Inf\{1 - // \tfrac{1}{2}(x + y)// : x, y \varepsilon BC(0), // x - y// \geq \varepsilon\}$.

We now give an important characteristic of the modulus of convexity in the following proposition.

<u>Proposition</u> **(3.0)**

The modulus of convexity of a Banach space x is a non-decreasing convex function.

Proof

The proof that $\delta x(\varepsilon)$ is non-decreasing is a trivial consequence of definition (2.0) and it is therefore omitted. To proof convexity, suppose for any two vectors U, V ε X, we denote by N(U,V) the set of all pairs x, ,y ε X with x, y ε B_1 (0), such that for some real scalars α_1, B_1 we have

$$x - y = \alpha U$$

$$x + y = BV$$

That is N(U,V) = $\{(x, y) : x, y \varepsilon B_1(0) \text{ and } x - y = \alpha U, x + y = BV\}$

For $r\varepsilon(0,2)$, define

$\delta(U,V,r) = \inf\{1 - \frac{1}{2} // x + y // ...., \mathrm{x, y}\varepsilon \mathrm{N}(\mathrm{U,V}), // \mathrm{x - y} // \geq \mathrm{r}\}$..................(1.4)

It is easy to see that $\delta(U,V,r) = 0$ for in (1.4)

$\{Since // \mathrm{x} // = 1, \text{for all } \mathrm{x}\varepsilon\mathrm{N}(\mathrm{U,V})\}$

Moreover, of r, for given any λ_1, λ_2 in (0,2) and $\varepsilon > 0$, we can choose $(\mathrm{x}_k y_k)\varepsilon N(\mathrm{U,V})$ such that $(\text{for } \mathrm{K} = 1,2)$

$$// \mathrm{x}_k, \mathrm{y}_k // \geq \lambda_k \text{...(1.5)}$$

and

$\delta(\mathrm{U,V},\lambda_k) + \frac{\varepsilon}{2} \geq 1 - \left(\frac{1}{2} // \mathrm{x}_k + y_k //\right)$.................................(1.6)

The choice of (x_k, y_k) is possible because of the definition $\delta(\mathrm{U,V},\varepsilon)$ in (1.5) as infimum.

Now, for $\lambda\mathrm{x} = (0,1)$

Let $\mathrm{x}_3 = \lambda\mathrm{x}_1 + (1-\lambda)\mathrm{x}_2$

And $y_3 = \lambda y_1 + (1-\lambda)y_2$

We have $// \mathrm{x}_3 // \leq \lambda // \mathrm{x}_1 // + (1-\lambda) // \leq 1 Cas\ \mathrm{x}_1, \mathrm{x}_2\, \overline{B_1(0)}$

Similarly

Also, $(\mathrm{x}_k, y_k)\varepsilon\mathrm{N}(\mathrm{U,V})$ implies that exist constants α_k, β_k $(k = 1,2)$ such that

$$\mathrm{x}_k - y_k = \alpha_k U$$

and $\mathrm{x}_k - y_k = \beta_k V$

From (1.6), we have

$$\begin{aligned}
\mathrm{x}_3 - y_3 &= \lambda\mathrm{x}_1 + (1-\mathrm{y})\mathrm{x}_2 + \lambda y_1 - (1-y)y_2 \\
&= \lambda(\mathrm{x}_1 - \mathrm{y}_1) + (1-\mathrm{y})(\mathrm{x}_2 - y_2) \\
&= \lambda\alpha_1 U + (1-y)\alpha_2 U, \qquad \text{from ....(1.7)} \\
&= //\lambda\alpha_1 + (1-\mathrm{y})\alpha_2 // U
\end{aligned}$$

If we set q= $\lambda\alpha_1 + (1-\lambda)\alpha_2, q$ some real number,

We have
$$\begin{aligned}
\mathrm{x}_3 - y_3 &= \lambda\mathrm{x}_1 + (1-\mathrm{y})\mathrm{x}_2 + \lambda y_1 - (1-y)y_2 \\
&= \lambda(\mathrm{x}_1 + y_1) + (1-\lambda)(\mathrm{x}_2 + y_2) \\
&= \lambda\beta_1 V + (1-\lambda)\beta_2 V \qquad \text{.....................(1.8)} \\
&= //\lambda\beta_1 + (1-\lambda)\beta_2 // V
\end{aligned}$$

So that for some real number $\gamma = \lambda\beta_1 + (1-\lambda)\beta_2$

We have

$$\begin{aligned}
&= \lambda\alpha_1 + (1-\lambda)\alpha_2 // U // \\
&= // \lambda\alpha_1 + (1-\lambda)\alpha_2 // U // \\
// \mathrm{x}_3 - y_3 // &= \lambda\alpha_1 // U // + (1-\lambda)\alpha_2 // U // \\
&= \lambda // \alpha_1 U // + (1-\lambda) // \alpha_2 U // \qquad \text{……………………………. (1.9)}
\end{aligned}$$

So that we have

$$= \lambda\beta_1 + (1-\lambda)\beta_2 \,//\, v \,//$$

$$- // x_3 + y_3 // \quad = // \lambda\beta_1 + (1-\lambda)\beta_2 \,//\, v \,//$$

$$= // \beta_1 v + (1-\lambda) // \beta_2 v$$

$$= \lambda // x_1 + y_1 // + (1-\lambda) // x_2 + y_2 // \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.10)$$

Now making use of (3.3.9), we get

$$// x_3 - y_3 // \geq \lambda\varepsilon_1 (1-\lambda)\varepsilon_2$$

But then (1.7) and (1.8) give

$$\delta(u, v, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) \leq 1 - \tfrac{1}{2} // x_3 + y_3 //$$

$$= 1 - \tfrac{1}{2} // \lambda // x_1 + y_1 // + (1-\lambda) // x_2 + y_2 //$$

$$= 1 - \lambda/2 // x_1 + y_1 // - \tfrac{1}{2} + (1-\lambda) // x_2 + y_2 //$$

$$= \lambda\left(1 - \tfrac{1}{2}\right) // x_1 + y_1 // + (1-\lambda)\left(1 - \tfrac{1}{2} // x_2 + y_2 //\right)$$

$$\leq \lambda(u, v, \lambda\varepsilon_1) + \tfrac{8}{2} // + (1-\lambda) // \delta(u, v, \lambda\varepsilon_2 + \varepsilon/2 // \quad \ldots\ldots\ldots\ldots (1.11)$$

From (1.0)

$$= \lambda\delta(u, v, \lambda\varepsilon_1) + (1-\lambda)\delta(u, v, \varepsilon_2) + \varepsilon/2$$

Now since ε is arbitrary, we infer that

$$= \delta(u, v, \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) \leq \lambda\delta(u, v, \varepsilon_1) + (1-\lambda)\delta(u, v, \varepsilon_2)$$

Thus, (u, v, ε) is convex. Now from the definitions of N(u, v) and δ (u, v, ε) each pair (x, y) $\varepsilon\, \overline{B}\,\overline{(0)} \times \overline{B}\,\overline{(0)}$.

Belong to some N(u, v), so that we have

$\delta\, x(\varepsilon) = Inf\{\delta(u, v, \varepsilon) : U, V_\varepsilon \varepsilon X, u \neq 0, v \neq 0$ and as $\delta(u, v, \varepsilon)$ is convex, So is $\delta\, x(\varepsilon)$

For the next proposition, we need the following lemma.

Lemma (2.0)

Suppose $f : (0,2) \to (0,1)$ is non-decreasing convex function and $0 \leq u < x < y \leq 2$ then

$$\frac{f(u) - f(v)}{v - u} \leq \frac{f(v) - f(x)}{y - x}$$

Proof

We can choose $\alpha\beta_\varepsilon (0,1)$ such that

$$V = \alpha u + (1-\alpha)y \quad \text{and}$$

$$U = \beta v + (1-\beta)y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.11)$$

$$\frac{f(v) - f(u)}{(v - u)} = \frac{f // \alpha u + (1-\alpha)y}{\alpha u + (1-\alpha)y - u} \quad \text{from (1.11)}$$

$$\leq \frac{\alpha f(u) + (1-\alpha) f(y) - f(u)}{\alpha u + (1-\alpha)y - u} \quad \text{as if is}$$

$$\frac{(1-\alpha) f(y) - (1-\alpha) f(u)}{(1-\alpha)(y - u)}$$

$$= \frac{f(y) - f(u)}{y - u} = \frac{\beta(f(y) - f(u))}{\beta(y - u)}$$

$$\frac{\beta f(y) - \beta f(u)}{y - x} \quad \text{from (1.11)}$$

$$= \frac{f(y) - \beta f(u) + \beta f(y) - f(y)}{y - x}$$

That is,

$$\frac{f(v) - f(u)}{v - u} \leq \frac{f(y) - // \beta f(u)}{y - x} + (1 - \beta) + (1 - \beta) f(y) // \text{...............(1.12)}$$

But since f is convex we get from (1.11)

Hence, (1.12) leads to

$$\frac{f(v) - f(u)}{v - u} \leq \frac{f(y) - f(x)}{y - x}$$

We now prove the following proposition

Proposition (3.0)

Let $f : (0,2) \to (0,1)$ be a non-decreasing convex function with f(0) = 0. Then for each x>2, f is continuous and has Lipschitz constant $(2\text{-}x)^{-1}$

Proof

As any point in the domain of f, r $\varepsilon(0,2)$, say is finite and f is non-decreasing then at each x>2, the right derivative f^1 r(x) of f exists (See Rockfella).

Thus, for v<x<2, we have by definition f^1 r(v+)

$$\lim_{x \to v} \frac{f(x) - f(v)}{x - v} \leq \lim_{x \to v} \frac{f(2) - f(x)}{2 - x} \qquad \text{by Lemma (1.11)}$$

$$= \frac{f(2) - f(v)}{2 - v}$$

$$= \frac{(1 - \alpha) f(y) - (1 - \alpha) f(u)}{(1 - \alpha)(y - u)}$$

$$\frac{f(y) - f(u)}{y - u} = \frac{\beta(f(y) - f(u))}{\beta(y - u)}$$

$$\frac{\beta(f(y) - \beta f(u))}{y - x} \qquad \text{from above}$$

$$= \frac{f(y) - f(u) + f(y)}{y - x} \quad - f(y) \text{ ....(1.13)}$$

That is,

$$\frac{f(v) - f(u)}{v - u} \leq \frac{f(y) - // \beta f(u)(1 - \beta) f(y) //}{y - x} \text{...............(1.14)}$$

But since f is convex, we get from (1.12)

$$f(x) = f // \beta u + (1 - \beta) y // \leq \beta f(u) + (1 - \beta) f(y) //$$

Hence, (3.3.12) leads to

$$\frac{f(v) - f(u)}{v - u} \leq \frac{f(y) - f(x)}{y - x}$$

We now prove the following proposition

Proposition (4.0)

Let $f : (0,2) \to (0,1)$ be a non-decreasing convex function with f(0) = 0. Then for each x>2, f is continuous and has Lipschitz constant $(2\text{-}x)^{-1}$

That is $= f^{1}r(v) \leq \frac{f(2)-f(v)}{2-v} \leq \frac{1}{2 - v}$

Replacing v by x, we have

$$= f^{1}r(v) \leq \frac{1}{2 - x}$$

Now, by the mean value theorem, for all x,y $\varepsilon(0,2)$ we have $// f(x) - f(y)// \leq // f(\varepsilon)// x - y//$ for some $\varepsilon(x,y)$

$$\leq \frac{1}{2 - x} \quad //x - y// \text{ from (A)}$$

That is $// f(x) - f(y)// \leq (2-x)^{-1} \;)// x - y//$

So that the function f is Lipschitz constant $(2\text{-}x)^{-I}$ on each interval (0,x).

The following proposition is also of interested

Proposition **(5.0)**

The Banach space x is uniformly convex if and only if $\delta x(\varepsilon) > 0$ for all $_{\varepsilon}\varepsilon(0,2)$

Proof

Suppose x is uniformly convex, then if $x, y\varepsilon X$ with $//x// \leq 1, //y// \leq 1$ and $//x - y// \geq \varepsilon$ we have

$$// \tfrac{1}{2}(x+y)// \leq 1 - \delta, \text{ for } \delta > 0$$

Now , $\delta x(\varepsilon) = Inf\ (1 - //x + y//, //x - y// \geq \varepsilon, //x// \leq 1, //y// \leq 1.$

$$\leq //1 - (1-\delta)// = \delta > 0$$

Let $\delta x(\varepsilon) > 0$ for $r\varepsilon(0,2)$ and choose $x, y\varepsilon x$ such that

$$// x // \leq 1, //y // \leq 1 \text{ and } //x - y // \geq \varepsilon$$

By definition

$$\delta x(\varepsilon) = Inf\left\{1 - // \tfrac{1}{2}(x+y)// : x, y\varepsilon \overline{B}\,\overline{(0)} \text{ and } //x - y// \geq \right\} > 0$$

This implies that for all $x, y\varepsilon \overline{B}\,\overline{(0)}$ with

$$// x - y// \geq \varepsilon, 1 - // \tfrac{1}{2}(x+y)\delta x(\varepsilon)$$

This implies that there exists a $\delta > 0$ such that

$$1 - // \tfrac{1}{2}(x+y)// \leq 1 - \delta, \delta > 0$$

Which implies uniform convexity

4.0 **NORMAL STRUCTURE AND REFELEXIVITY OF BANACH SPACES**

In this section, we deal with some other geometric properties which are important in studying the fixed point theory of non-expansive mapping.

Let C be a bounded convex subset of a Banach spacex.

The diameter d of c is define by

D = Sup{// zi-zj //, zj, ziCA}

A point Zo$\in X$ is said to be a diameter point of C

If Sup $\{// z_0 - z // : z \in\} = d$

And a point $z^* \in x$ is called a non-diamental point of c

If Sup $\{// z^* - z // : z \in -C\} < \text{Sup}\,\{//zi - zj// : zi, zj \in C\}$

Definition 3.0

A bounded convex subset of Banach space is said to have normal structure if for each convex subset x of C, consisting of more than one point A, contains a non-diamental point that is there exists a $z_0 \in A$.

Such that Sup $\{// z_0 - z // : z \in C\} < \text{Sup}\ \{//\text{zi - zj}// : \text{zi, zj} \in \text{A}\}$

Geometrically, C is said to have normal structure if for every convex, C subset A of C there exist a ball whose radius is less than diameter of A centered at a point of A which contains A.

In the following, we exhibit large classes of spaces with normal structure.

Proposition (6.0)

Every uniformly convex set in X, containing as least two different point Z_1, Z_2. Suppose ∂ is the diameter of C and $z_0 = \frac{1}{2}(Z_1 + Z_2)$ for any $Z \in C$, proposition (1.2) given us, from

$$// z - z_1 // \le \partial; //Z_1 - Z_2 // \le \partial \qquad = //Z - Z_0 // \le \frac{\{1 - \delta(// Z_1 - Z_2 //)\}}{0}$$

$$\tfrac{1}{2}\{(z - Z_1) + (Z - Z_2)\} = \tfrac{1}{2}\{(Z - Z_2)\}$$

Since $\qquad = z - z_0$

The above inequality implies that C is contained in the ball of radius, say, r, less than ∂ centered at Z_0 , that is, $Z \in B(Z_0, r) \rightarrow C \subseteq B_r(Z_0)$.

Where $r = Sup\{// z_0 - z // : z \in C\}$

Proposition (7.0)

Every convex and impact subset of a Banach space normal structure.

Proof

We proof this by this contradiction that is, we shall assume that a compact convex subset of C of a Banach space X does not have normal structure. Then we shall generate a sequence, which will contradict the hypothesis of compactness.

Suppose C does not have normal structure, then we may assume that all point of C are diamental for C. let Z_0 be the diameter of C, we shall construct a sequence Z, Z_2,………… Of point of C that

$$// Z_i - Z_j // = d \ \ (\text{i, j} = 1,2,\ldots\ldots\ldots \text{i} \ne \text{j})$$

To do this, we choose $Z_1 \in C$ arbitrary and assume that Z_2,Z_3,………………….Z_n have already been chosen.

By the convexity of C

$\frac{1}{n}(Z_1 + \ldots\ldots\ldots\ldots\ldots + Z_n)$ is a point of C and thus by assumption, is diamental for C. by the compactness of C, the "Sup" is achieved in C so that we can find a point $Z, Z_{n+1} \in C$ such that

$$\frac{// Z_{n+1} - Z_1 + \ldots\ldots\ldots\ldots + Z_n //}{n} = \text{d}$$

Consequently, $//Z_{n+1}-Z_j// = d$ for j =1..,…

Which means that the sequences /$\{Z_n\}$ n≥ 1 has no convergent subsequence, and thus has no cluster value in C. this contradicts the compactness of C and completes the proof.

We observe that if a convex set C has normal structure, then so does every convex subset of C. in particular, if the whole space X has normal structure, then every convex subset of X has normal structure. This follows from the definition.

Some Banach space does not have the following.

Example

1. The Space X = C(0,1) with Sup norm does not have normal structure.
2. The space X = L_1 $(0,2\pi)$ does not have normal structure.
3. The space X = I_1 does not have normal structure. Is called a retracting mapping.

Definition **(3.4.10)**

X is a retract of Y, if XCY and there exist a continuous mapping α.

Lemma **(3.4.10)**

Let M be a closed convex subset of a Hilbert space. If T is a non-expansive mapping of M into H, then I-T is a restriction to M of a monotone operator.

Proof

Let r be the metric retraction of H into H, for $x, y \in H$.

$$\begin{aligned}
&(I\text{-}Tr)x - (I\text{-}TR), x\text{-}y) \\
&= //x\text{-}y//^2 - (Trx\text{-}Try, x\text{-}y) \\
&\geq // x - y//^2 - //Trx - Try// //x - y// \\
&\geq 0
\end{aligned}$$

I-Tr is monotone

The retraction to M I – Tr is I – T

Lemma **(3.4.10)****

If T is monotone and X_0 and Y_0 are normal of H such that Tx = Y_0

Proof

For any Y in H and T>0

Let $y_t = X_0 + ty_0$

With y- y_t then $(Ty_t - y_0, y) \geq 0$ so that

$$(Ty_t - y_0, y) \geq (y_0 Tx_0, y)$$

Let $t \to 0^+$, then $Ty_t \to Ty_0$

So that $(y_0 Tx_0, y)$

Then Y_0 = Tx

(3.4.11) **Conjectures**

(1) Let M be a be a convex subset of a normal linear space L let T be a non-expansive mapping on M into L, Then for 0<t<1, the mapping St = tI(1- t)T is non-expansive and the same set of fixed points is T.

If TMCM, then St <Cm

In fact, by kransnoseleki, we have the following conjectives.

(2). Consider a mapping

$T : M \to L$, Where T is non-expensive, L is a normed linear space MCL and convex, then if $G = \frac{1}{2}(I + T)$, G is non-expansive and the exists $X^* \varepsilon M$ such that $GX^* = X^*$

Also, $X_{n+1} = \lambda_{xm} + (1-\lambda)T_{xn}$

$\lambda \varepsilon (0,1)$ is true in a uniformly convex space.

Conclusion

We have examined in this paper project, the non-expansive mapping and the fixed point theorem. We have also been able to show that classes Banach of spaces with normal structures are examples of uniformly convex spaces. Thus, the Lp space $(1 < p < \infty)$ are classical examples of uniformly convex spaces.

Therefore, is application a non-expensive operator will have solution on these spaces.

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