On Simple And Bisimple Left Inverse Semi Groups

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ABSTRACT: This paper deals with Simple and bi-simple inverse semi-groups. The general properties and characteristics of simple and bi-simple semi- groups and inverse semi-groups were discussed. [Researcher. 2010;2(8):57-67]. (ISSN: 1553-9865).

Key words: Simple, bi-simple semigroup, inverse semigroup

INTRODUCTION


The importance of simple and bi-simple semi group in Algebra has developed into area of independent research of Mathematics, [1 - 13] have come to play a very significant role. In this paper, we look in to simple and bi-simple left inverse semi groups.

According to Encyclopaedia of mathematics, A semi-group not containing proper ideals or congruences of some fixed type. Various kinds of simple semi-groups arise, depending on the type considered: ideal-simple semi-groups, not containing proper two-sided ideals (the term simple semi-group is often used for such semi-groups only); left (right) simple semi-groups, not containing proper left (right) ideals; (left, right) simple semi-groups, semi-groups with a zero not containing proper non-zero two-sided (left, right) ideals and not being two-

element semi-groups with zero multiplication; bi-simple semi-groups, consisting of one $D$-class (cf. Green equivalence relations); bi-simple semi-groups, consisting of two $D$-classes one of which is the null class; and congruence-free semi-groups, not having congruences other than the universal relation and the equality relation.

Every left or right simple semi-group is bi-simple; every bi-simple semi-group is ideal-simple, but there are ideal-simple semi-groups that are not bi-simple (and even ones for which all the $D$-classes consist of one element).

Various types of simple semi-groups often arise as "blocks" from which one can construct the semi-groups under consideration. For classical examples of simple semi-groups see Completely-simple semi-group; Brandt semi-group; Right group; for bi-simple inverse semi-groups (including structure theorems under certain restrictions on the semi-lattice of idempotents) see [1], [8], [9]. There are ideal-simple inverse semi-groups with an arbitrary number of $D$-classes. In the study of imbedding of semi-groups in simple semi-groups one usually either indicates conditions for the possibility of the corresponding imbedding, or establishes that any semi-group can be imbedded in a semi-group of
the type considered. E.g., any semi-group can be
imbedded in a bi-simple semi-group with an identity
(cf. [1]), in a bi-simple semi-group generated by
idempotents (cf. [10]), and in a semi-group that is
simple relative to congruences (which may have
some property given in advance: the presence or
absence of a zero, completeness, having an empty
Frattini sub-semi-group, etc., cf. [3]–[5]).

2. SIMPLE SEMIGROUP: A semigroup S is said to be simple if it contains only one \( J - \) class and BISIMPLE
SEMIGROUP: A semigroup S is said to be bisimple if it contains only one \( D - \) class.

Theorem 1.0

(i) If a \( D - \) class of a semigroup S contains a regular
element, then every element of \( D \) is regular. \( J - \)
(ii) If \( D \) is regular then every \( L - \) class and \( R - \) class contained in \( D \) contain an idempotent.

Proof:
An element of a semigroup S is regular if and only if \( R_a [L_b] \) contains an idempotent.
It then follows that an \( R - \) class \( R - (L - \) class) contains a regular element; then it contain an idempotent and
every element of \( R (L) \) is regular.
Since every \( R - \) class of S contained in \( D \) meet every \( L - \) class of S contained in \( D \), then (1) holds.

Lemma 1.0
If \( a \) and \( a^{-1} \) are inverse element of a semigroup S.
Then \( e = aa^{-1} \) and \( f = a^{-1}a \) are idempotent such
that \( ea = af = a \) and \( a^{-1}e = f a^{-1}a = a^{-1} \)
Hence \( e \in R_a \cap L_a^{-1} \) and \( f \in R_a^{-1} \cap L_a \). The element \( a, a^{-1}, e \) and \( f \) all belong to the same \( D - \) class.

Theorem 1.1
Let \( 'a' \) be a regular element of a semigroup S
Then (i) Every inverse of \( a \) lies in \( D_a \)
(ii) An \( H - \) class, \( H_b \) contains an inverse of \( a \) if and only if both \( R \) of the \( H - \) classes \( R_a \cap L_b \) and \( R_b \cap L_a \)
contains idempotent.
(iii) No \( H - \) class contains more than one inverse of \( a \)

COROLLARY 1.0
A semigroup S is an inverse semigroup if and only if each
\( L - \) class and each \( R - \) class contains exactly one idempotent.

Corollary 1.1
A semigroup S is simple if and only if
\( S \) a \( S = S \) \( \forall \ a \in S \ i.e \ if \ and \ only \ if \)
\( \forall a, b \in S \ \exists \ x, y \in S \ such \ that \ xay = b \)

Theorem 1.2
The following statements about a semigroup S are equivalent.
(i) \( S \) is an inverse semigroup
(ii) \( S \) is regular and idempotent element commute.
(iii) Each \( L - \) class and \( R - \) class of S contain a unique idempotent.
(iv) Each principal left and right ideal of S contains a unique idempotent generator.

Proof:
It is clear by the definition of \( L \) and \( R \) that III and IV are equivalent.
To show that I $\Rightarrow$ II
Let $e, f$ be idempotent and let $x = (ef)^{-1}$
Then, $ef x e = ef$ and $xefx = x$
The element $fxe$ is idempotent since
$(fxe)^2 = f(xefx) = fxe$
Also, $(ef) (fxe) (ef) = efxf = ef$
$(fxe) (ef) (fxe) = f(xefx) = fxe$
and $ef$ is an inverse of $fxe$.
But $fxe$ being idempotent, it is its own unique inverse.
So $fxe = ef$.
It is then follows that $ef$ is idempotent and
similarly we obtain that $fe$ is also idempotent
Hence, $(ef) (fe) (ef) = (ef)^2 = ef$
$(fe) (ef) = (fe)^2 = fe$
$\therefore$ $fe$ is an inverse of $ef$.
But $ef$, being an idempotent is its own unique inverse and so., $ef = fe$
$\therefore$ I $\Rightarrow$ II.
To show that II $\Rightarrow$ III
Since $S$ is regular every $\mathcal{L}$ – class contain at least one idempotent.
If $e, f$ are $\mathcal{L}$- equivalent idempotent, then $ef = e, fe = f$.
Since by hypothesis, $ef = fe$, it follows that $e = f$
Similarly remark apply to $\mathcal{R}$ – class we can express the property III of inverse semigroups as follows.
$\mathcal{L} \cap (EXE) = \mathcal{R} \cap (EXE) = 1E$
Where $E$ is the set of idempotent of $S$.
$\therefore$ II $\Rightarrow$ III.
To show that III $\Rightarrow$ I
Since a semigroup with the property III is necessarily regularly, then every $\mathcal{D}$ – class contains an idempotent.
If $a_{1}, a_{11}$ are inverse of $a$, then $aa_{1}$ and $aa_{11}$ are idempotent in $S$ that are $\mathcal{R}$ equivalent to ‘$a$’ and hence to each other.
By property III, we have $aa' = aa''$
Equally, $a'a = a''a$ and so
$a' = a'aa' = a''aa' \Rightarrow a''aa'' = a''$

**Proposition 1.0**
Let $S$ be an inverse semigroup with semilattice of idempotent $E$.
Then (i) $(a')' = a \ \forall \ a \ in \ S$.
(ii) $e' = e \ \forall \ e \ in \ E$
(iii) $(ab)' = b' a' \ \forall \ a, b \ in \ S$
(iv) $aa' e \ in \ E, a' e a \ in \ S$ and $\forall e$ in $E$
(v) $a, b$ if and only if $aa'' = bb''$
(a) $\mathcal{L}$ $b$ if and only if $a'a = b''b$
(vi) If $e, f$ in $E$, then $e, f$ in $\mathcal{D}$, $f$ in $S$ if and only if $\exists$ a in $S$ such that $aa'' = e, a'a = f$.

**Proof:**
I and II are mutuality of the inverse property of a semigroup
To proof III
Since $bb^{-1}$ and $a^{-1}a$ are idempotent
$(ab)(b^{-1}a^{-1}) = a(bb^{-1})(a^{-1}b)$
$= aa^{-1}abb^{-1}b$
$= ab$.
Also, $(b^{-1}a^{-1})(ab)(b^{-1}a^{-1}) = b^{-1}(a^{-1}a)(bb^{-1}) a^{-1}$
Thus $b^{-1}a^{-1}$ is an inverse and hence the inverse of $ab$.  

i.e $(ab)^{-1} = b^{-1}a^{-1}$

To proof IV

$(aea^{-1})^2 = ae(a^{-1}a)ea^{-1}$

$= aa^{-1}ae^{-1}a^{-1} = aea^{-1}$

Similarly, $(a^{-1}ea)^2 = a^{-1}ea$.

Recall that a semigroup $S$ is said to be simple if it contains only one $\mathcal{J}$-class. i.e $S$ is simple if and only if $\mathcal{J} = S \times S$

i.e every element in $\mathcal{J}$ is related to each other.

**Lemma 1.2**

An inverse semigroup $S$ with semilattice of idempotent $E$ is simple if and only if

$(\forall e, f \in E)(\exists g \in E) [g \leq f \text{ and } e \mathcal{D} g]$

**Proof:**

Let $S$ be simple, if $e, f \in E$, then $e \mathcal{J} f$ and so $\exists x, y \in S$ such that $e = xfy$.

Let $g = fyx$, then

$g^2 = fyx(fyx)ex = fyx^2x$

since $g = fyx$, $g \in E$

Also,

$Fg = g$ and $g \leq f$

If $z = x^2e$, then $xz = xx^2e = xx^2xfy$

$\Rightarrow xfy = e$ and so $e \mathcal{L} z$

Also

$Zx = x^{-1}ex = x^{-1}e^2x$

$= x^{-1}xfyxex = x^{-1}xxy = gx^{-1}x$

$= fyx^2x = fyx = g$

$gx^{-1} = gx^{-1}xx^{-1} = x^{-1}xgx^{-1}$

$= x^{-1}xfyx^2x^{-1} = x^{-1}e^2xx^{-1}$

$\Rightarrow x^{-1}xx^{-1}e = x^{-1}e = x^{-1}e = z$ and so $z \mathcal{R} g$.

Thus, $e \mathcal{D} g$ as required.

Conversely if $S$ has the property described above, considering any two idempotent $e$, $f$ in $S$ then $\exists g \in E : g \leq f$

and $e \mathcal{D} g$ and so, $J_e = J_g < J_f$.

Equally, $\exists h \in E : h \leq e$ and $f \mathcal{D} h$ and so, $J_f = J_h \leq J_e$.

Hence,$J_e = J_f$ and so all the idempotent of $S$ fall in a single $\mathcal{J}$-class.

But every element of $S$ in $\mathcal{J}$ – equivalent (indeed even $\mathcal{R}$ – or $\mathcal{L}$ – equivalent) to some idempotent and so it follows that $S$ is simple.

As a consequence if $S$ is a simple inverse semigroup with semilattice of idempotent $E$, then $E$ has the property $(\forall e, f \in E)(\exists g \in E) [g \leq f \text{ and } Ee \sim Eg]$.

Recall that a semigroup $S$ is said to be Bisimple if it contains only one $D$-class. It is a semigroup in which $D$ is the universal relation.

If $S$ is a Bisimple inverse semigroup with semilattice of idempotent $E$, then all the idempotent are mutually $D$-equivalent. i.e $D \cap (E \times E) = E \times E$.

Hence it follows that $U = E \times E$, i.e $E$ is a uniform semilattice.

Conversely, if we start with a uniform semilattice $E$, then we cannot expect that every inverse semigroup having $E$ as semilattice of idempotent will be Bisimple, $E$ itself is one such inverse semigroup and are assumed not Bisimple.
DEFINITION
If \((e, f) \in U\),
Let \(T_{e, f}\) be the set of all isomorphism from \(E_e\) onto \(E_f\).
Let \(T_E = \bigcup_{e, f \in U} T_{e, f}\).
Since all the element of \(T_E\) are partial one-one mapping of \(E\). We may therefore multiply element of \(T_E\) as element of \(J(E)\).

If \(\alpha: E_e \to E_{e'}\) and
\(\beta: E_{e'} \to E_{e''}\) are element of \(T_{E'}\), then the product of \(\alpha\) and \(\beta\) in \(J_E\) maps \((E_{e'} \cap E_{e''})\alpha^{-1}\) onto \((E_{e''} \cap E_{e'}\)\) \(\beta\) i.e it maps \((E_{e''})\alpha^{-1}\) onto \((E_{e''})\beta\) \((f_{e''})\alpha^{-1}\) and \((f_{e''})\beta = j\).

Then \(x \in E (E_{e''})\alpha^{-1} \iff x \in E_{e''} \alpha^{-1}\)
i.e \(x \leq f_{e''} \iff x \leq (f_{e''})\alpha^{-1}\)
\(\Rightarrow x \in E_i\)
Similarly, \(x \in E (E_{e''})\beta\).
\(\Rightarrow x \in E_j\).
Thus \(\alpha \beta\) maps the principal ideal \(E_i\) onto the principal ideal \(E_j\).
Since it is clearly an isomorphism, we have that \(\alpha \beta \in T_E\). Thus \(T_E\) is a subsemigroup of \(J(E)\).

Proposition 1.2
If \(E\) is a uniform semilattice, then \(T_E\) is a Bisimple inverse semigroup.

Proof:
This proves more generally that if \(E\) is any semilattice whatever, then in \(T_E\) \(D \cap (E \times E) = U\).
Since \(T_E\) is an inverse semigroup whose semilattice of idempotent is (effectually) \(E\), one half of this result is obvious.
Suppose that \((e, f) \in U\). Then \(E_e \sim E_f\) and so there exist at least one \(\alpha\) in \(T_E\) such that \(\text{dom}(\alpha) = E_e\) and \(\text{ran}(\alpha) = E_f\). i.e \(\alpha \alpha^{-1} = 1_{E_e} (= e)\) and
\(\alpha^{-1} \alpha = 1_{E_f} (= f)\) and so \(e, f\) are \(D\)-equivalent in \(T_E\).
By applying it into the uniform case, we find that all idempotent in \(T_E\) are \(D\)-equivalent.
Hence since every element of a regular semigroup is \(D\)-equivalent \(T_E\) is therefore BISIMPLE

DEFINITION:
Let \(T\) be a semigroup with identity \(I\) and
Let \(\theta\) be a homomorphism from \(T\) into \(H_1\) the \(H\)-class containing the identity of \(T\) (what is often called the group of units of \(T\))
Let \(N = \{0, 1, 2, \ldots\}\).
We can make \(N \times T \times N\) into a semigroup by defining.
\((m, a, n) (p, b, q) = (m - n + t, a^{\theta t} b^{\theta t}, q - p + t)\)
where \(t = \max(\ n, p)\) and \(\theta^0\) is interpreted as the identity map of \(T\).
To check that the given composition is associative, we observe that:
\([(m, a, n) (p, b, q)] (r, c, s) = m - n + w, a^{\theta w} b^{\theta w}, q - p + t)\)
where \(U = \max(q - p + \max(n, p) r)\)
\(W = \max(n, p - q \max(q - r))\)
The outer coordinates in multiplication (**) combining exactly as in the bicyclic semigroup which associative since it is isomorphism to \(T_{cw}\).
Hence by equating the first coordinates or (equivalently third coordinates) we obtain
\(W = u + p - q\).
It is then clear that this result implies the quality of the two middle coordinates and so the composition (**) is indeed associative and shall be denoted by the semigroup obtained in this way by \(S = BR(T, \theta)\) which refers to as the BRUCK – Reilly Extension of \(T\) determined by \(\theta\).
Proposition 1.3

If \( T \) is a semigroup with identity 1 and \( S = BR(T, \theta) \).

Then,
(i) \( S \) is a simple semigroup with identify \((0, 1, 0)\)
(ii) Two element \((m, a, n)\) and \((p, b, q)\) of \( S \) are D- equivalent in \( S \) if and only if \( a \) and \( b \) are D- equivalent in \( T \).
(iii) The element \((m, n)\) of \( S \) is idempotent if and only if \( m = n \) and \( a^2 = a \).
(iv) \( S \) is an inverse semigroup if and only if \( T \) is an inverse semigroup.

Proof:

(i) We show that if \((m, a, n)\) and \((p, b, q)\) are arbitrary element of \( S \) the \( \exists (r, x, s) \) such that \((r, x, s) = (pp, b, q)\).

(1) Let \((r, u, s) = (p(a\theta)^{-1}m + 1)\) and \(t, y, u) = (n + 1, b, q)\) where \((a, \theta)^{-1}\) is the inverse of \(a\theta\) in the group \(H_i\).

Then it is easy to check that the desired equality holds.

(ii) Let us use superscripts \( S \) and \( T \) to distinctiuished between the green equivalent on \( S \) and those on \( T \). if \((m, a, n) R^s (p, b, q)\) then \( P = m – n + \max(n, r) \geq m\)

EQUALLY, we show that \( m \geq p \) and so infect \( m = p \) it follows that \( m – n + \max(n, r) = m \) and \( n \geq r \).

By equating the middle coordinate, we have \( A(x\theta^m)^x = b \), so \( R_b \leq R_s \) in \( T \).

Similarly, we show that \( R_s \leq R_b \) and so \( aR^Tb \).

Conversely, if \( aR^Tb \) then \( ax = b \), \( bx_1 = a \) for some \( x, x_1 \) in \( T \).

Hence,
\[
(m, a, n) (n, x, q) = (m, b, a)
\]

\[
(m, b, q) (q, x, n) = (m, a, n) \text{ in } S \text{ and so } (m, a, n) R^s (m, b, q)
\]

\[
\Rightarrow (m, a, n) R^s (p, b, q) \iff m = p \text{ and } aR^Tb
\]

A dual argument establishes that \((m, a, n) L^s (p, b, q) \iff n = q \) and \( aL^Tb \).

Suppose that \((m, a, n) \) and \( p, b, q \) in \( S \) are such that \((m, a, n) D^s (p, b, q) \).

It then follows that \( aR^Tc \) and \( cL^Tb \) (and \( r = m, s = q \)

Hence \( aD^Tb \).

Conversely, if \( aD^Tb \), then for some \( c \) in \( T \) we have \( aR^Tc \) and \( cL^Tb \).

Therefore for every \( m, n, p, q, \) in \( N \)

\[
(m, a, n) R^t (m, c, q) (m, c, q) L^t (p, b, q)
\]

and so \((m, a, n) D (p, b, q) \).

(iii) \((m, a, n)^2 = (m – n + t, a\theta^{im}b\theta^{jm}, n – m + t)\)

where \( t = \max(m, n) \)

Hence \((m, a, n)\) can be idempotent only if \( m = n \).

Since \((m, a, m)^2 = (m, a^2, m)\), the element \((m, a, m)\) is idempotent if and only if \( a^2 = a \).

(iii) If \( T \) is an inverse semigroup, then each element \((m, n)\) of \( S \) has an inverse \((n, a^1, m)\)

Thus \( S \) is regular.

To show that it is an inverse semigroup.

Let \((m, e, m)\) \((n, f, n)\) be idempotent in \( S \) (with \( m \geq n \) say)

Then
\[
(m, e, m) (n, f, n) = (m, e(f\theta^{m-n}) m)
\]
\[
(n, f, n) (m, e, m) = (m(f\theta)^{m-n} e, m)
\]

Now \( f\theta^{m-n} \) is an idempotent in \( T \).

Indeed if \( m = n \), we must have \( f\theta^{m-n} = \theta \) the only idempotent in \( H_i \).

Hence \( e(f\theta^{m-n}) = (f\theta^{m-n}) e \) and so idempotent commutes in \( S \).
Conversely if $S$ is an inverse semigroup and if $(p, b, q)$ is the inverse of $(m, a, n)$, then $(m, a, n)(p, b, q) = (m – n + t, a \theta t – n, b \theta t – p, q – p + t)$ with $t = \max (n, p)$ is an idempotent $R^\ast$- equivalent to $(m, a, n)$ and $L^\ast$- equivalent to $(p, b, q)$.

Therefore $m = m – n + t = q – p + t = q$ and so $n = p (= t)$, $m = q$.

The inverse property now gives

$(m, a, n) = (m, a, n)(n, b, m)(m, a, n) = (maba, n)$

$(n, b, m) = (n, b, m)(m, a, n)(n, b, m) = (nbab, n)$

Thus $aba = a bab = a$ and so is an inverse of $a$ in $T$. Thus $T$ is regular.

If $e,f$ are idempotent in $T$, then the commuting of the idempotent $(o, e, o), (o, f, o)$ of $S$ implies that $ef = fe$ in $T$.

2.1 A semigroup $S$ is called left inverse if every principal right ideal of $S$ has a unique idempotent generator. Many authors and scholars have laid their hands in solving problems relating to simple and bisimple semigroup. Here in this chapter, we investigate the $D$- class of regular semigroups and of left inverse semigroups.

Lemma, proposition and Theorems were also considered to support each statement.

A description of a bisimple let inverse semigroup $S$ with identity element $e$ as a quotient of the contesian product $L, x L_0$ of $L – class L_0$ of and the $R – class R_0$ of $S$ containing $e$.

We also describe the maximal inverse semigroup homomorphism of $S$.

3. D – CLASSES IN REGULAR SEMIGROUPS

Let $S$ be a regular semigroup and $a \in S$. The $L –$ class of $S$ containing the element $a$ is demoted by $L_a$.

Let $A$ be a subset. Throughout $a'$ denotes an inverse of $a$ and $A'$ denotes the set of all inverse of elements of $A$.

**Lemma 1**

Let $S$ be a regular semigroup. Then $S$ is Bisimple if and only if for any two idempotent $e, f$ in $S$ there exist an element $a$ of $S$ and $a'$ of $a$ such that $aa' = f$.

**Lemma 2**

Let $S$ be a regular semigroup and $e$ be an idempotent of $S$ write $L = L_e, R = R_e, H = H_e$ and $D = D_e$, then

i. $L \subseteq R^1$ and $R \subseteq L^1$

ii. $LR = D$

iii. Let $m, n, \in L$ and $b, d \in R$. then $mb = nd$ if and only if $\exists u \in H$ such that $mu = n$ and $ud = b$.

**Proof:**

Let $x \in L$. then $\exists x'$ of $x$ such that $x' \in R$. so $x \in R'$ and

Hence $L \subseteq R'$

Similarly, Let $m, n, \in L$ and $b, d \in R$ then there exist inverse $m', n'$, $b'$ and $d'$ of $m, n, b, and d$ respectively such that $m'm = n'n = bb' = dd' = e$.

Let $md = nd$.

Then $(mm' nd) d'$

And so $mu'n = n$. Let $u = m'n$.

Now $mu = n$ and $ud = m'nd = m'mb = b$.

Further, $cu = uc = u$ and $u(d'b) = bb' = e = n'n = (n'm)u$.

Thus $u \in H$.

**Remark** The element $u$ above is unique.

If $X$ is a subset of a semigroup $S$. then $E(X)$ denote the set of all idempotent in $x$.

Let $S$ be a regular semigroup for any $a \in S$.

**Lemma 3**

Let $D$ be a $D$ – class of regular semigroup $S$.

Let $E(D)$ be a subsemigroup of $S$. Then $D$ is a bisimple subsemigroup of $S$.

**Proof:**

Let $a, b \in D$ and let $f = a'a$ and $g = bb'$

Then $f \in L_a$ and $g \in R_b$, so $fg \in L_a R_b$

But $L_a R_b$ is contained in some $D –$ class $D'$ of $S$.

Since $fg \in D$. By hypothesis we then conclude that $D' = D$ and $a \in L_a R_b$.
Hence, D is a subsemigroup of S

Lemma 4
Let S be a regular semigroup and e be an idempotent of S. Write \( L = L_e \), \( R = R_e \) and \( D = D_e \).

Let \( E(S) \) be a subsemigroup of S. Then the following conditions on S are equivalent:

i. \( f e f = f \) for any idempotent \( f \in D \)
ii. \( R \) is a subsemigroup of S
iii. \( L \) is a subsemigroup of S

Proof:
Assume (1). Let \( a, b \in R \). Then there exist inverses \( a' \) of \( a \) and \( b' \) of \( b \) such that \( aa' = bb' = e \).

By (1) we have \( a' a = a \) and \( a a' = e \).

That is \( bb'a' = e \), now \( b' a' \) is an inverse of \( ab \) and therefore \( ab \in R \).

Conversely, Assume (ii)
Let \( f^2 = f = D \). Then there exist \( a \in R \) and an inverse \( a' \) of \( a \) such that \( aa' = e \) and \( a'a = f \). By Hypothesis \( ef \) and \( fe \) are idempotent.

Therefore \( ea' \) is an inverse of \( ae \).

By (ii) we have \( a e \in R \)

Hence \( aea' \in R \). Now \( ef = aeea = a' (aea)e \)

\( = a'(ea) = a'a = f \) given A.

Thus (I) and (II) are equivalent

Similarly (I) and (III) are also equivalent

Therefore (I) \( \Rightarrow \) (II) \( \Rightarrow \) (III) in an arbitrary regular semigroup S.

Lemma 5
Let S be a regular semigroup and e be an idempotent of S. Write \( L = L_e \), \( R = R_e \), and \( D = D_e \).

Let \( e \) be a left or right identity element for \( D \).

Then \( R \) and \( L \) are subsemigroups of S.

Proof:
Let \( a, b, \in R \), then there exist inverses \( a' \) of \( a \) and \( b' \) of \( b \) such that \( aa' = bb' = e \). As \( e \) is a left or right identity for \( D \) we get \( peq = pq \) or any \( p, q, \in D \).

Now \( abb'a = aea = a' = e \) and so \( b'a \) is an inverse of \( ab \). Hence \( ab, e \in R \) and \( R \) is a subsemigroup of S.

Similarly \( L \) is a subsemigroup of S.

Lemma 6
Let S be a regular semigroup and e be an idempotent of S. Write \( L = L_e \), \( R = R_e \), \( D = D_e \).

The following conditions on S are equivalent:

(I) \( e \) is a right [left] identity element for \( D \).

(II) \( e \) is an identity element for \( R \) [\( L \)].

(III) \( R \) [\( L \)] is a right [left] cancellative subsemigroup of S.

Proof:
Clearly (I) implies (II). Conversely assume (II).

Let \( f^2 = f \in D \), let \( a \in R \cap L_e \).

Then there exist an inverse \( a' \) of \( a \) such that \( a'a = f \).

Now by (II) we get \( fe = aae = a'a = f \).

So we get (I). Hence (I) and (II) are equivalent.

Now assume (I). Then by Lemma 5.3 \( R \) is a subsemigroup of S.

Let \( ax = bx \) where \( a, b, x \in R \). As \( xx' = e \)

For some inverse \( x' \) of \( x \), by (I) we get \( a = b \), and hence (III)

Assume (III) let \( a \in R \), then \( a e \in R \) now \( aee = ae \) and by right cancellative we get \( ae = a \) so we get (II) and hence (I).

Lemma 7
Let $S$ be a regular semigroup and $e$ be an idempotent of $S$. Write $R = R_e$ and $D = D_e$.

(I) $Sa \cap R \subseteq Ra$ for any $a \in R$

(II) if $R$ is a subsemigroup of $S$ then $Sa \cap R \subseteq R_a$ for any $a \in R$

(III) Let $e$ be an identity element for $D$ and $a \in R$, then $Sa \cap R = R_a$ if and -only if $a \in R$

Proof:

(I) Let $a \in R$ and $x \in Sa \cap R$, then $x = ta \in R$ for some $t \in S$. Now there exist inverse $a'$ of $a$ and $(ta)'$ of $ta$ such that $aa' = (ta)' = e$.

So $e = t(ea)(ta)' \in E \subseteq S$.

Again $ta = (te)a$ given $te = e \in S$.

Hence $te \in R$. Now $x = ta = (te)a \in R_a$ proving (I).

(II) If $R$ is a subsemigroup of $S$, then $R_a \subseteq R$ for any $a \in R$. Now from (I) we get (II)

(III) Let $e$ be an identity for $D$. Then from Lemma (5.1 and (2) above we get $Sa \cap R = R_a$ for any $a \in R$. The converse is obvious

4 D - CLASSES IN LEFT INVERSE SEMIGROUP

Recall that a semigroup $S$ is called a left (right) inverse semigroup if every principal right (left) ideal of $S$ has unique idempotent generator.

A left (right) inverse semigroup is clearly a regular semigroup.

Lemma 1

Let $S$ be a regular semigroup. Then the following condition on $S$ are equivalent.

(I) $e \cap S = eS = ee$ for any two idempotent $e, f$ in $S$.

(II) $ef = ef$ for any two idempotent $e, f$ in $S$.

(III) If $a'$ and $a''$ are inverse of $a$ in $S$, then $aa' = aa''$.

(IV) $S$ is a left inverse semigroup.

Corollary 1

Let $S$ be a left inverse semigroup and $e$ be an idempotent of $S$. Then

(I) $aa' = e$ for any inverse $a'$ of $a$ in $Re$.

(II) $E(S)$ is a subsemigroup of $S$.

(III) If $a', b'$ are inverse of $a, b$ in $S$ then $b'a'$ is an inverse of $ab$.

Theorem 1

Let $S$ be regular semigroup. Then $S$ is a left inverse semigroup if and only if $L_e = (R_e)'$ for any idempotent $e$ in $S$.

Proof:

Let $e$ be any idempotent in $S$ write $L = L_e$ and $R = R_e$.

Let $S$ be a left inverse semigroup and $P \in R^1$.

Then $p = X'$ is an inverse of some $X \in R$.

Now $xx'$ is an idempotent in $Re$.

Hence $xx' = e$ since $S$ is left inverse.

Consequently $x' \in L xx' = L_e$ and hence $R' \in L$.

Conversely, let $L = R'$ for any idempotent $e$ in $S$.

Let $f$ and $g$ be idempotent of $S$ and let $fs = gs$. then $gf = f$, $fg = g$ and $f$ is an inverse of $g$.

Now by hypothesis we get $f \in Lg$.

So $fg = f$ and hence $f = g$.

Thus $S$ is a left inverse semigroup.

Lemma 2

Let $S$ be a left inverse semigroup and $e$ be an idempotent of $S$. Let $a, c, u, e \in R_e$.
Let \( a', c' \) be inverse of \( a, c \) respectively, then

(I) \[ if \ a' u = c', then \ a = uc \]

(II) \[ if \ a' = c', then \ a = c. \]

**Proof:**

(i) \[ let \ a' u = c, \] Let \( u' \) be an inverse of \( u. \)

Then \( aa' = uu' = e \) and \( a = c' u. \)

Therefore \( a'a = a' (uc) \)

This implies that \( a = uc. \)

(ii) \[ Let \ a' = c', \] then \( a \) and \( c \) both are inverse of \( a' \)

\[ \therefore \ a'a = a' (uc) \]

Hence \( a = c. \)

**Lemma 3**

Let \( S \) be a left inverse semigroup and \( e \) be an idempotent of \( S. \)

Let \( e \) be an identity element for \( D_e. \)

Let \( c, d \in R = Re \) then \( Rc = Rd \) if and only if for any given inverse \( c' \) of \( c, \) there exist an inverse \( d' \) of \( d \) such that \( c' c = d' d. \)

**Proof:**

Let \( Rc = Rd \) and \( Let \ c' \) be the given inverse of \( c. \)

Now \( c = id \) and \( d = je \) for some \( i, j \in R. \)

Also \( cc' = dd'' = e \) for any inverse \( d' \) of \( d \)

So \( e = cc' = (i, j, c) c' = ij \) and

Similarly, \( e = ji. \)

Now \( c' i \) is an inverse of \( d = je \) and \( d'd = c' id = c. \)

**Theorem 2**

Let \( D \) be a \( D – \) class of the left inverse semigroup \( S. \)

Let \( R \) be an \( R- \) class of \( S \) contained in \( D \) then.

The following condition on \( S \) are equivalent.

(I) \[ E(D) \] is a subsemigroup of \( S. \)

(II) \[ D \] is a (Bisimple) subsemigroup of \( S. \)

(III) \[ For \ any \ a, b, \in R there exist c \in R such that \ Sa \cap Sb = Sc. \]

**Proof:**

(i) \[ Implies (II) by lemma 5.3. \]

Now Assume (II), \( Let \ a, b, \in R. \)

Let \( a' \) be an inverse of \( a \) and \( b' \) an inverse of \( b. \)

Let \( a'a = f \) and \( b'b = g, \) then \( f, g \) and \( fg \in D \)

\[ \therefore \ Sa \cap Sb = Sf \cap Sg = Sfg \] by lemma 5.6

Let \( c \in R \cap Lfg. \) Then, \( Sfg = Sc. \)

Assume (III):

\[ \text{Let} \ f, g \in E(D). \]

Let \( a \in R \cap Lf \) and \( b \in R \cap Lg. \)

Then by lemma 5.6, \( Sa \cap Sb = Sfg. \)

But there exist \( c \in R \) such that \( Sfg = Sc \)

\[ \therefore \ fg \in D, \text{ so } fg \in E(D). \]
Conclusion
We only focused on the D-classes of left inverse semigroup whereby we established that a left inverse semigroup is clearly a regular semigroup.

References