On Simple And Bisimple Left Inverse Semi Groups

[1]Oladejo, N.K.; Makanjuola , S.O and [3]Adetunde, I.A.
[1]Department of Applied Mathematics and Computer Science
University for Development studies Navrongo, Ghana
[2]Department of Mathematics, University of Ilorin. Ilorin, Nigeria.
[3]Department of Mathematics, University of Mines and Technology
Tarkwa, Ghana
adetunde@googlemail.com

Prof. Isaac Adetunde,
University of Mines and Technology,
Faculty of Engineering ,
Dept.of Mathematics.
P.O.Box 237,
Tarkwa.Ghana.
Website: www.isaacadetunde.com
Ghana Tel:+233-24-3151871,+233-27-5032707,
+233-20-7112264, +233-24-6101306
Nigeria Tel: +2348033777786, +2348055676596, +2348057071943

ABSTRACT: This paper deals with Simple and bi-simple inverse semi-groups. The general properties and characteristics of simple and bi-simple semi- groups and inverse semi-groups were discussed. [Researcher. 2010;2(8):57-67]. (ISSN: 1553-9865).

Key words: Simple, bi-simple semigroup, inverse semigroup

INTRODUCTION

Relevant publications on Simple and Bi-Simple are the work done by A.H. Clifford and G.B. Preston (1961–1967), E.S. Lyapin (1974), V.N. Klimov (1973), R. Baer and F. Levi (1932), K. Byleen, J. Meakin, and F. Pastijn (1978), F. Pastijn (1977), E.G. Shutov (1963), W.D. Munn (1969), J.M. Howie (1976). to mention but few. Recently Junghenn (1996) worked on Operator semigroup compactifications.

The importance of simple and bi- simple semi group in Algebra has developed into area of independent research of Mathematics, [1 - 13] have come to play a very significant role. In this paper, we look in to simple and bi- simple left inverse semi groups.

According to Encyclopaedia of mathematics, A semi-group not containing proper ideals or congruences of some fixed type. Various kinds of simple semi-groups arise, depending on the type considered: ideal-simple semi-groups, not containing proper two-sided ideals (the term simple semi-group is often used for such semi-groups only); left (right) simple semi-groups, not containing proper left (right) ideals; (left, right) 0-simple semi-groups, semi-groups with a zero not containing proper non-zero two-sided (left, right) ideals and not being two-element semi-groups with zero multiplication; bi-simple semi-groups, consisting of one $\mathcal{D}$-class (cf. Green equivalence relations); 0-bi-simple semi-groups, consisting of two $\mathcal{D}$-classes one of which is the null class; and congruence-free semi-groups, not having congruences other than the universal relation and the equality relation.

Every left or right simple semi-group is bi-simple; every bi-simple semi-group is ideal-simple, but there are ideal-simple semi-groups that are not bi-simple (and even ones for which all the $\mathcal{D}$-classes consist of one element).

Various types of simple semi-groups often arise as "blocks" from which one can construct the semi-groups under consideration. For classical examples of simple semi-groups see Completely-simple semi-group; Brandt semi-group; Right group; for bi-simple inverse semi-groups (including structure theorems under certain restrictions on the semi-lattice of idempotents) see [1], [8], [9]. There are ideal-simple inverse semi-groups with an arbitrary number of $\mathcal{D}$-classes. In the study of imbedding of semi-groups in simple semi-groups one usually either indicates conditions for the possibility of the corresponding imbedding, or establishes that any semi-group can be imbedded in a semi-group of the type considered. E.g., any semi-group can be imbedded in a bi-simple semi-group with an identity (cf. [1]), in a bi-simple semi-group generated by idempotents (cf. [10]), and in a semi-group that is simple relative to congruences (which may have some property given in advance: the presence or absence of a zero, completeness, having an empty Frattini sub-semi-group, etc., cf. [3]–[5]).

2. SIMPLE SEMIGROUP: A semigroup S is said to be simple if it contains only one $\mathcal{J}$ – class and **BISIMPLE SEMIGROUP:** A semigroup S is said to be bisimple if it contains only one $\mathcal{D}$ – class.

Theorem 1.0

(i) If a $\mathcal{D}$ – class of a semigroup S contains a regular element , then every element of $\mathcal{D}$ is regular. $\mathcal{J}$ -)

(ii) If $\mathcal{D}$ is regular then every $\mathcal{L}$ - class and $\mathcal{R}$ – class contained in $\mathcal{D}$ contain an idempotent.

Proof:

An element of a semigroup S is regular if and only if $\mathcal{R}_a$ [$\mathcal{L}_a$] contains an idempotent.

It then follows that an $\mathcal{R}$ – class $\mathcal{R}$- ($\mathcal{L}$ – class) contains a regular element ; then it contain an idempotent and every element of $\mathcal{R}$ ($\mathcal{L}$) is regular.

Since every $\mathcal{R}$ – class of S contained in $\mathcal{D}$ meet every $\mathcal{L}$ – class of S contained in $\mathcal{D}$, then (1) holds.

Lemma 1.0

If a and a^{-1} are inverse element of a semigroup S.
Then $e = aa^{-1}$ and $f = a^{-1}a$ are idempotent such that $ea = af = a$ and $a^{-1}e = f\, a^{-1}a = a^{-1}$

Hence $e \in \mathcal{R}_a \cap \mathcal{L}_{a^{-1}}$ and $f \in \mathcal{R}_{a^{-1}} \cap \mathcal{L}_a$. The element a, a^{-1}, e and f all belong to the same $\mathcal{D}$ – class.

Theorem 1.1

Let ‘a’ be a regular element of a semigroup S

Then (i) Every inverse of a lies in $\mathcal{D}_a$

(ii) An $\mathcal{H}$- class, $\mathcal{H}_b$ contains an inverse of a if and only if both of the $\mathcal{H}$ – classes $\mathcal{R}_a \cap \mathcal{L}_b$ and $\mathcal{R}_b \cap \mathcal{L}_a$ contains idempotent.

(iii) No $\mathcal{H}$ – class contains more than one inverse of a

COROLLARY 1.0

A semigroup S is an inverse semigroup if and only if each $\mathcal{L}$ – class and each $\mathcal{R}$ – class contains exactly one idempotent.

Corollary 1 .1

A semmigroup S is simple if and only if
$SaS = S \;\; \forall\, a \in S$ i.e if and only if
$\forall a, b \in S \;\ni\; x , y$ in S such that $xay = b$

Theorem 1.2

The following statements about a semigroup S are equivalent.

(i) S is an inverse semigroup
(ii) S is regular and idempotent element commute.
(iii) Each $\mathcal{L}$ – class and $\mathcal{R}$ – class of S contain a unique idempotent.
(iv) Each principal left and right ideal of S contains a unique idempotent generator.

Proof:

It is clear by the definition of $\mathcal{L}$ and $\mathcal{R}$ that III and IV are equivalent.

To show that I $\Rightarrow$ II
Let e, f be idempotent and let $x = (ef)^{-1}$
Then, ef x ef = ef and xefx = x
The element fxe is idempotent since
$(fxe)^2 = f(xefx) = fxe$
Also, (ef) (fxe) (ef) = efxef = ef
(fxe) (ef) (fxe) = f(xefx) = fxe
and ef is an inversed of fxe.
But fxe being idempotent, it is its own unique inverse.
So fxe = ef.
It is then follows that ef is idempotent and
similarly we obtain that fe is also idempotent
Hence, (ef) (fe) (ef) = $(ef)^2$ = ef
(fe) (ef) (fe) = $(fe)^2$ = fe
:. fe is an inverse of ef.
But ef, being an idempotent is its own unique inverse and so,, ef = fe
:. I $\Rightarrow$ II.
To show that II $\Rightarrow$ III
Since S is regular every $\mathcal{L}$ – class contain at least one idempotent.
If e, f are $\mathcal{L}$- equivalent idempotent, then ef = e, fe = f.
Since by hypothesis, ef = fe, it follows that e = f
Similarly remark apply to $\mathcal{R}$ – class we can express the property III of inverse semigroups as follows.
$\mathcal{L} \cap (EXE) = \mathcal{R} \cap (EXE) = 1E$
Where E is the set of idempotent of S.
:. II $\Rightarrow$ III.
To show that III $\Rightarrow$ I
Since a semigroup with the property III is necessarily regularily, then every $\mathcal{D}$ – class contains an idempotent.
If $a,^1$ a^{11} are inverse of a, then aa^1 and aa^{11} are idempotent in S that are $\mathcal{R}$ equivalent to 'a' and hence to each other.
By property III, we have $aa' = aa''$
Equally, $a'a = a''a$ and so
$a' = a'aa' = a''aa'$
$\Rightarrow a''aa'' = a''$

Proposition 1.0
Let S be an inverse semigroup with semilattice of idempotent E.
Then (i) $(a^{-\prime})^{-\prime} = a \;\; \forall$ a in S.
(ii) $e^{-\prime} = e \;\; \forall$ e in E
(iii) $(ab)^{-\prime} = b^{-\prime}a^{-\prime} \;\; \forall$ a, b in S
(iv) $aea^{-\prime} \in E$, $a^{-\prime}ea \in E$, $\forall$a in S and $\forall$e in E
(v) a $\mathcal{R}$ b if and only if $aa^{-\prime} = bb^{-\prime}$
a $\mathcal{L}$ b if and only if $a^{-\prime}a = b^{-\prime}b$
(vi) If e, f $\in$ E, then e $\mathcal{D}$, f in S if and only if $\ni$ a in S such that $aa^{-\prime} = e$, $a^{-\prime}a = f$.

Proof:
I and II are mutuality of the inverse property of a semigroup
To proof III
Since bb^{-1} and $a^{-1}a$ are idempotent
$(ab)(b^{-1}a^{-1})(ab) = a(bb^{-1})(a^{-1}a)b$
$= aa^{-1}abb^{-1}b$
= ab.
Also, $(b^{-1}a^{-1})(ab)(b^{-1}a^{-1}) = b^{-1}(a^{-1}a)(bb^{-1})a^{-1}$

$= b^{-1}bb^{-1}a^{-1}aa^{-1}$
$= b^{-1}a^{-1}$
Thus $b^{-1}a^{-1}$is an inverse and hence the inverse of ab.
i.e $(ab)^{-1} = b^{-1}a^{-1}$
To proof IV
$(aea^{-1})^2 = ae\,(a^{-1}a)\,ea^{-1}$
$= aa^{-1}ae^2 a^{-1} = aea^{-1}$
Similarly, $(a^{-1}ea)^2 = a^{-1}ea$.

Recall that a semigroup S is said to be simple if it contains only one $\mathcal{J}$- class. i.e S is simple if and only if $\mathcal{J}$ = S X S

i. e every element in $\mathcal{J}$ is related to each other.

Lemma 1.2
An inverse semigroup S with semilattice of idempotent E is simple if and only if
$(\forall\, e, f \in E)(\ni g \in E)\,[g \leq f \text{ and } e\, \mathcal{D})g]$.

Proof:
Let S be simple, if e, f $\in$ E, then e J_f and so $\exists$ x, y $\in$ S such that e = xfy.
Let g = fyex, then
$g^2 = fye(xfy)ex = fye^3x$
since g = fyex, g $\in$E
Also,
Fg = g and $g \leq f$
If $z = x^{-1}e$, then $xz = xx^{-1}e = xx^{-1}xfy$
$\Rightarrow xfy = e$ and so $e\, \mathcal{L}\, z$
Also
$Zx = x^{-1}ex = x^{-1}e^2x$
$= x^{-1}xfyex = x^{-1}xy = gx^{-1}x$
$= fyexx^{-1}x = fyex = g$
$gx^{-1}= gx^{-1}xx^{-1} = x^{-1}xgx^{-1}$
$= x^{-1}\, xfyexx^{-1} = x^{-1}\, e^2\, xx^{-1}$
$\Rightarrow x^{-1}xx^{-1}e = x^{-1}e = x^{-1}e = z$ and so $z\, \mathcal{R}\, g$.

Thus, e $\mathcal{D}$ g as required.
Conversely if S has the property described above, considering any two idempotent e, f in S then $\ni g \in E : g \leq f$ and e $\mathcal{D}$ g and so, $J_e = J_g < J_f$.
Equally, $\ni h \in E: h \leq e$ and f $\mathcal{D}$ h and so, $J_f = J_h \leq J_e$.
Hence.
$J_e = J_f$ and so all the idempotent of S fall in a single
$\mathcal{J}$ – class.

But every element of S in $\mathcal{J}$ – equivalent (indeed even $\mathcal{R}$ – or $\mathcal{L}$ – equivalent) to some idempotent and so it follows that S is simple.
As a consequence,if S is a simple inverse semigroup with semilattice of idempotent E, then E has the property $(\forall\, e, f \in E)(\ni g \in E)\,[g \leq f \text{ and } Ee \sim Eg]$.
Recall that a semigroup S is said to be Bisimple if it contains only one D-class. It is a semigroup in which D is the universal relation.

If S is a Bisimple inverse seigroup with semilattic of idempotent E. then all the idempotent are mutually D-equivalent. i.e $D \cap (E \times E) = E \times E$.

Hence it follows that U = E x E, i.e E is a uniform semilattice.

Conversely, if we start with a uniform semilattice E, then we cannot expect that every inverse semigroup having E as semilattiec of idempotent will be Bisimple, E itself is one such inverse semigroup and are assumed not Bisimple.

DEFINTION

If (e, f) ∈ U,

Let T_e, f be the set of all isomorphism from E_e onto E_f

Let $T_E = \bigcup_{e,\, f \in u} T_e, f$

Since all the element of T_E are partial one-one mapping of E. We may therefore multiply element of T_E as element of $J_{(E)}$.

If $\alpha: E_e \to E_f$ and

$\beta: E_g \to E_h$ are element of T_E, then the product of α and β in J_E maps $(E_f \cap E_g)\alpha^{-1}$ onto $(E_g \cap E_g)\ \beta$ i.e it maps $(E_{fg})\alpha^{-1}$ onto $(E_{fg})\beta\ (f_g)\alpha^{-1}$ and $(f_g)\beta = j$.

Then $x\ E\ (E_{fg})\alpha^{-1} \Leftrightarrow x\ \alpha \in E_{fg}$

i.e $x\ \alpha \leq f_g \Leftrightarrow x \leq (f_g)\alpha^{-1}$

$\Rightarrow x \in E_i$

Similarly, $x\ E\ (E_{fg})\ \beta$.

$\Rightarrow x\ E\ E_j$.

Thus α β maps the principal ideal E_i onto the principal ideal E_j.

Since it is clearly an isomorphism, we have that $\alpha\ \beta \in T_E$. Thus T_E is a subsemigroup of $J_{(E)}$.

Proposition 1.2

If E is a uniform semilattice, then T_E is a Bisimple inverse semmigroup.

Proof:

This proves more generally that if E is any semilattice whatever, then in T_E $D \cap (E \times E) = U$.

Since T_E is an inverse semigroup whose semilattice of idempotent is (effectually) E, one half of this result is obvious.

Suppose that (e,f) E U. Then $E_e \sim E_f$ and so there exist at least one α in T_E such that dom(α) = E_e and ran(α) = E_f. i.e $\alpha\alpha^{-1} = 1E_e$ (= e) and

$\alpha^{-1}\alpha = 1E_f$ (= f) and so e,f are D – equivalent in T_E.

By applying it into the uniform case, we find that all idempotent in T_E are D- equivalent.

Hence since every element of a regular semigroup is D- equivalent T_E is therefore BISIMPLE

DEFINITION:

Let T be a semigroup with identify I and

Let θ be a homomorphism from T into H_i the H-class containing the identity of T (what is often called the group of units of T)

Let N = { 0, 1, 2---}.

We can make N x T x N into a semigroup by defining.

$(m, a, n)\ (p, b, q) = (m - n + t,\ a\theta^{t-n}\ b\theta^{t-p},\ q - p + t)$

where t = max(n, p) and θ^0 is interpreted as the identity map of T.

To check that the given composition is associative, we observe that:

$[(m, a, n)\ (p, b, q)]\ (r, c, s) = m - n + w,\ a\theta^{w-n}\ b\theta^{w-r-p+q},\ s - r - + q + w)$

where

$$\left.\begin{aligned} U &= \max(q - p + \max(n, p)\, r) \\ W &= \max(n, p - q \max(q - r)) \end{aligned}\right\} \quad \cdots \qquad **$$

The outer coordinates in multiplication (**) combining exactly as in the bicyclic semigorp which associative since it is isomorphism to T_{cw}.

Hence by equating the first coordinates or (equivalently third coordinates) we obtain

W = u + p – q.

It is then clear that this result implies the quality of the two middle coordinates and so the composition (**) is indeed associative and shall be denoted by the semigroup obtained in this way by S = BR(T, θ) which refers to as the BRUCK – Reilly Extension of T determined by θ.

Proposition 1.3

If T is a semigroup with identity 1 and S = BR (T, θ).

Then,

(i) S is a simple semigroup with identify (0, 1, 0)

(ii) Two element (m, a, n) and (p, b, q) of S are D- equivalent in S if and only if a and b are D – equivalent in T.

(iii) The element (m a, n) of S is idempotent if and only if m = n and $a^2 = a$.

(iv) S is an inverse semmigroup if and only if T is an inverse semigroup.

Proof:

(1) We show that if (m, a, n) and (p,, b, q) are arbitrary element of S the ∃ (r, x, s) and (t, y, u) such that (r, x, s) (m, a, n) (t, y, u) = (pp, b, q).

(1) Let (r, u, s) = $(p(a\theta)^{-1}m + 1)$ and

(t, y, u) = (n + 1, b, q) where

$(a, \theta)^{-1}$ is the inverse of aθ in the group H_i

Then it is easy to check that the desired equality holds.

That (0, 1, 0) is the identify of S is a matter of routine verification.

(ii) Let us use superscripts S and T to distriquished between the green equivalent on S and those on T. if (m a, n) R^s (p, b, q) for some (r x, s) in S

Hence P = m – n + max (n, r) ≥ m

Equally, we show that m ≥ p and so infect m = p it follows that m – n + max (n, r) = m and

Hence that n ≥ r.

By equating the middle coordinate, we have

$A(x\theta^{n-r} = b$, so $R_b \le R_a$ in T.

Similarly, we show that $R_a \le R_b$ and so aR^Tb.

Conversely, if aR^Tb then ax = b, $bx^1 = a$ for some x, x^1 inT^1).

Hence,

(m, a n) (n, x, q) = (m, b, a)

(m, b, q) (q, x, n) = (m, a n) in S and so (m, a, n) R^s(m, b q)

⇒ (m, a, n) R^s (p, b, q) ⇔ mm = p and aR^Tb

A dual argument establishes that (m, a, n) L^s (p, b, q) ⇔ n = q and a L^Tb.

Suppose that (m, a, n) and p b, q) in S are such that

(m, a, n) D^s (p, b, q). Then there exist (r, c, s) in S for which (m, a, n) R^s (r, c s) L^s (p, b, q).

It then follows that aR^Tc and cL^Tb (and r = m, s = q)

Hence a D^Tb.

Conversely, if a D^Tb, then for some c in T we have aR^Tc and cL^Tb .

Therefore for every m, n p, q, in N

(m, a, n) R^s (m, c q) (m, c, q) L^s (p, b, q) and so (m, a n) D (p, b, q).

(iii) $(m, a, n)^2 = (m – n + t, a\theta^{t-m} b\theta^{t-m}, n – m + t)$

where t = max (m, n)

Hence (m, a, n) can be idempotent only if m = n.

Since $(m, a, m)^2 = (m, a^2, m)$, the element (m, a, m,) is idempotent if and only if $a^2 = a$.

(iii) If T is an inverse semigroup, then each element (m, a n) of S has an inverse (n, a^{-1}, m)

Thus S is regular.

To show that it is an inverse seigroup.

Let (m, e, m) (n, f, n) be idempotent in S (with m ≥ n say)

Then

$$\left.\begin{array}{l}(m,e,m)(n,f,n)=(m,e(f\theta^{m-n})\,m)\\(n,f,n)(m,e,m)=(m(f\theta)^{M-n}e,m\end{array}\right\}$$

Now $f\theta^{m-n}$ is an idempotent in T.

(indeed if m ≠ n, we must have $f\theta^{m-n}$ = 1the only idempotent in H_i)

Hence e $(f\theta^{m-n}) = (f\theta^{m-n})$ e and so idempotent commutes in S.

Conversely if S is an inverse semigroup and if (p, b, q) is the inverse of (m, a, n), then (m, a, n) (p, b, q) = (m – n + t,
$a\theta^{t-n} b\theta^{t-p}$, q – p + t) with t = max (n, p) is an idempotent R^s – equivalent to (m,, a n) and L^s – equivalent to (p, b, q).
Therefore m = m – n + t = q – p + t = q
and so n = p (= t) , m, = q.
The inverse property now gives
(m, a, n) = (m, a, n)(n, b m) (m, a, n) = (m aba, n)
(n, b, m) = (n,, b, m) (m, a, n)(n, b, m) = (n, bab, n)
Thus aba = a bab = a and so is an inverse of a in T. Thus T. is regular.
If e,f are idempotent in T, then the commuting of the idempotent (o, e, o), (o, f, o) of S implies that $e_f = f_e$ in T.

2.1 A semigroup S is called left inverse if every principal right ideal of S has a unique idempotent generator. Many authors and scholars have laid their hands in solving problems relating to simple and bisimple semigroup. Here in this chapter, we investigate the D- class of regular semigroups and of left inverse semigroups.

Lemma, proposition and Theorems were also considered to support each statement.

A description of a bisimple let inverse semigroup S with identity element **e** as a quotient of the contesian product L_ex L_e of L – class L_e of and the R – class R_e of S containing **e**.

We also describe the maximal inverse semigroup homomorphism of S.

3. D – CLASSES IN REGULAR SEMIGROUPS

Let S be a regular semigroup and **a** $\in$ S. The L – class of S containing the element **a** is demoted by L_a.
Let A be a subset. Throughout a' denotes an inverse of a and A' denotes the set of all inverse of elements of A

Lemma 1

Let S be a regular semigroup. Then S is Bisimple if and only if for any two idempotent e, f in S there exist an element a of S and a' of a such that $\mathbf{aa^1}$ = f.

Lemma 2

Let S be a regular semigroup and e be an idempotent of S write L = L_e, R = R_e, H = H_e, and D = D_e, then

i. $L \subseteq R^1$ and $R \subseteq L^1$
ii. LR =D
iii. Let m, n, $\in$ L and b, d $\in$R. then mb = nd if and only if $\exists$ u $\in$ H such that mu = n end ud = b.

Proof :

Let x $\in$ L. then $\exists$ x′ of x such that x′ $\in$ R. so x $\in$ R′ and
Hence L $\subseteq$ R′
Similarly, Let m, n, $\in$ L and b, d $\in$R then there exist inverse m′, n′, b′ and d′ of m, n, b, and d respectively such that m′m = n′n = bb′ = dd′ =e.
Let md = nd.
Then (mm′ nd) d′
And so mu′n = n. Let u = m′n.
Now mu = n and ud = m′nd = m′mb =b.
Further, eu = ue = u and u(d′b) = bb′ = e = n′n = (n′m)u.
Thus u $\in$ H.

Remark The element u above is unique.
If X is a subset of a semigroup S. then E (X) denote the set of all idempotent in x.
Let S be a regular semigroup for any a $\in$S.

Lemma 3

Let D be a D – class of regular semigroup S
Let E (D) be a subsemigroup of S. Then D is a bisimple subsemigroup of S

Proof:

Let a, b, $\in$ D and let f = a′a and g = bb′
Then f $\in L_a$ and g $\in R_b$, so f g $\in L_a R_b$.
But $L_a R_b$ is contained in some D – class D′ of S.
Since fg $\in$D. By hypothesis we then conclude that D' = D and a b $\in L_a R_b$

Hence, D is a subsemigroup of S

Lemma 4

Let S be a regular semigroup and **e** be an idempotent of S. Write $L = L_e$, $R = R_e$ and $D = D_e$

Let E (S) be a subsemigroup of S. Then the following condition on S are equivalent

i. f e f = f for any idempotent f $\in$D

ii. R is a subsemigroup of S

iii. L is a subsemigroup of S

Proof:

Assume (1). Let a, b $\in$ R. then there exist inverse a′ of a and b′ of b such that aa′= bb′ =e

By (1) we have a′ a e a′ a = a′a and aea′ = aa′aa′ = e

That is abb′a′ = e, now b′ a′ is an inverse of ab and therefore ab $\in$R.

Conversely Assume (ii)

Let f^2 = f = D. then exist a $\in$R and an inverse a′ of a such that aa′ =e and a′a = f. by Hypothesis ef and fe are idempotent.

Therefore ea′ is an inverse of ae.

By (ii) we have a e $\in$R

Hence aea′ $\in$R. Now fef =a′aea′a = a′ (aea′e)a

= a′ea = a′a = f given A.

thus (1) and (II) are equivalent

similarly (I) and (III) are also equivalent

therefore 1$\Rightarrow$ (II) $\Rightarrow$ (III) in an arbitrary regular semigroup S

Lemma 5

Let S be a regular semigroup and **e** be an idempotent of S write $L = L_e$, $R = R_e$ and $D = D_e$

Let **e** be a left or right identity element for D.

Then R and L are subsemigroup of S.

Proof:

Let a, b, $\in$ R, then there exist inverse a′ of a and b′ of b such that aa′ = bb′ = e. As e is a left or right identity for D we get peq = pq or any p, q, $\in$D.

Now abb′a = aea′ = aa′ = e and so b′a′is an inverse of ab. Hence ab, $\in$ R and R is a subsemigroup of S. Similarly L is a subsemigroup of S.

Lemma 6

Let S be a regular semigroup and **e** be an idempotent of S.

Write $L = L_{e,}$ $R = R_{e,}$ $D = D_{e..}$ The following conditions on S are equivalent .

(I) **e** is a right [left] identity element for D.

(II) **e** is an identity element for R [L]

(III) R [L] is a right [left] cancellative subsemigroup of S.

Proof:

Clearly (I) implies (II). Conversely assume (II).

Let f^2 = f $\in$ D. let a$\in$ R $\cap$ L_f.

Then there exist an inverse a′a of a such that a′a = f.

Now by (II) we get fe = a′ae = a′a = f

So we get (I). Hence (I) and (II) are equivalent.

Now assume (I). Then by lemma 5.3 R is a subsemigroup of S.

Let ax = bx where a, b, x$\in$ R. As xx′ = e

For some inverse x′ of x, by (I) we get a = b. and hence (III)

Assume (III) let a $\in$ R, then a e $\in$ R now aee = ae and by right cancellative we get ae =a so we get (II) and hence (I).

Lemma 7

Let S be a regular semigroup and **e** be an idempotent of S. Write R = R_e and D = D_e.

(I) Sa $\cap$ R $\subseteq$ Ra for any a $\in$ R

(II) if R is a subsemigroup of S then Sa $\cap$ R $\subseteq$ R_a for any a $\in$ R

(III) Let **e** be an identity element for D and a $\in$ R, then
S_a $\cap$ R = Ra if and -only if a $\in$ R

Proof:

(I) Let a $\in$ R and x $\in$ Sa $\cap$ R, then x = t a $\in$ R for some t $\in$ S. Now there exist inverse a′ of a and (ta) ′ of ta such that aa′ = ta (ta) ′ = e.

So e = t (ea) (ta) ′ $\in$ E e S.

Again ta = (te) a given te = e t e $\in$ S.

Hence te $\in$ R. now x = ta = (te) a $\in$ R_a proving (I).

(II) If R is a subsemigroup of S, then R_a $\in$ R for any
a $\in$ R. now from (I) we get (II)

(III) let **e** be an identify for D. then from lemma (5.1 and (2) above we get $S_a \cap R = R_a$
For any a $\in$ R. the converge is obvious

4 D - CLASSES IN LEFT INVERSE SEMIGROUP

Recall that a semigroup S is called a left (right) inverse semigroup if every principal right (left) ideal of S has unique idempotent generator.

A left (right) inverse semigroup is clearly a regular semigroup.

Lemma 1

Let S be a regular semigroup. Then the following condition on S are equivalent.

(I) $S_e \cap S_f$ = S_{ef} (=S_{ef}) for any two idempotent e,f in S.

(II) fef = ef for any two idempotent e f in S.

(III) If a′ and a′′ are inverse of a in S , then aa′ = aa′′

(IV) S is a left inverse semigroup

COROLLARY 1

Let S be a left inverse semigroup and **e** be an idempotent of S. Then

(I) aa′ = e for any inverse a′ of a in Re.

(II) E (S) is a subsemigroup of S.

(III) If a′, b′ are inverse of a, b in S then b′a′ is an inverse of ab.

Theorem 1

Let S be regular semigroup. Then S is a left inverse semigroup if and only if $L_e = (R_e)'$ for any idempotent **e** in S.

Proof:

Let **e** be any idempotent in S write L = L_e and R = R_e

Let S be a left inverse semigroup and P$\in R^1$

Then p = X′ is an inverse of some X$\in$R.

Now xx′ is an idempotent in Re.

Hence xx′ = e since S is left inverse.

Consequently x′ $\in$ L xx′= L_e and hence R′ $\in$L.

Conversely, let L = R′ for any idempotent **e** in S.

Let f and g be idempotent of S and let fs = gs. then gf = f, fg = g and f is an inverse of g.

Now by hypothesis we get f $\in$ Lg.

So fg = f an hence f = g.

Thus S is a left inverse semigroup.

Lemma 2

Let S be a left inverse semigroup and **e** be an idempotent of S. Let a, c, u, $\in$, R_e.

Let a',c' be inverse of a, c respectively, then

(I) if $a'u = c'$, then $a = uc$

(II) If $a' = c'$, then $a = c$.

Proof:

(i) let $a'u = c$, Let u' be an inverse of u.

Then $aa' = uu' = e$ and $a' = c'u'$.

Therefore $a'a = a'(uc)$

This implies that $a = uc$.

(ii) Let $a' = c'$, then a and c both are inverse of a'

$\therefore$ $a'a = a'(uc)$

Hence $a = c$.

Lemma 3

Let S be a left inverse semigroup and **e** be an idempotent of S.

Let **e** be an identity element for D_e.

Let $c, d \in R = R_e$ then $Rc = Rd$ if and only if for any given inverse c' of c, there exist an inverse d' of d such that $c'c = d'd$.

Proof:

Let $Rc = Rd$ and Let c' be the given inverse of c.

Now $c = id$ and $d = jc$ for some $i, j \in R$.

Also $cc' = dd'' = e$ for any inverse d'' of d

So $e = cc' = (i,j, c)\, c' = ij$ and

Similarly, $e = ji$.

Now $c'i$ is an inverse of $d = jc$ and $d'd = c'id = c'c$.

Theorem 2

Let D be a D – class of the left inverse semigroup S.

Let R be an R- class of S contained in D then.

The following condition on S are equivalent.

(I) E(D) is a subsemigroup of S.

(II) D is a (Bisimple) subsemigroup of S.

(III) For any $a, b, \in R$ there exist $c \in R$.
such that $Sa \cap Sb = Sc$.

Proof:

(i) Implies (II) by lemma 5.3.

Now Assume (II), Let $a, b, \in R$.

Let a' be an inverse of a and b' an inverse of b.

Let $a'a = f$ and $b'b = g$. then f, g and $f\,g \in D$

$\therefore$ $Sa \cap Sb = Sf \cap Sg = Sfg$ by lemma 5.6

Let $c \in R \cap Lfg$. Then, $Sfg = Sc$.

Assume (III):

Let $f, g \in E(D)$.

Let $a \in R \cap Lf$ and $b \in R \cap Lg$.

Then by lemma 5.6, $Sa \cap Sb = Sfg$.

But there exist $c \in R$ such that $Sfg = Sc$

$\therefore$ $fg \in D$, so $fg \in E(D)$.

Conclusion

We only focused on the D-classes of left inverse semigroup whereby we established that a left inverse semigroup is clearly a regular semigroup.

References

(1) A.H. Clifford, G.B. Preston, "Algebraic theory of semi-groups" , **1–2** , Amer. Math. Soc. (1961–1967)
(2) E.S. Lyapin, "Semigroups" , Amer. Math. Soc. (1974) (Translated from Russian)
(3) L.A. Bokut', "Some embedding theorems for rings and semigroups" *Sibirsk. Mat. Zh.* , **4** : (1963) pp. 500–518 (In Russian)
(4) E.G. Shutov, "Embeddings of semigroups in simple and complete semigroups" *Mat. Sb.* , **62** : 4 (1963) pp. 496–511 (In Russian)
(5) V.N. Klimov, "Embedding of semigroups in factorizable semigroups" *Sib. Math. J.* , **14** : 5 (1973) pp. 715–722 *Sibirsk. Mat. Zh.* , **14** : 5 (1973) pp. 1025–1036
(6) R. Baer, F. Levi, "Vollständige irreduzibele Systeme von Gruppenaxiomen" *Sitzungsber. Heidelb. Akad. Wissenschaft. Math.-Nat. Kl.* , **2** (1932) pp. 3–12
(7) M. Teissier, "Sur les demi-groupes admettant l'existence du quotient d'un cote" *C.R. Acad. Sci. Paris* , **236** : 11 (1953) pp. 1120–1122
(8) W.D. Munn, "Some recent results on the structure of inverse semigroups" K.W. Folley (ed.) , *Semigroups* , Acad. Press (1969) pp. 107–123
(9) J.M. Howie, "An introduction to semigroup theory" , Acad. Press (1976)
(10) F. Pastijn, "Embedding semigroups in semibands" *Semigroup Forum* , **14** : 3 (1977) pp. 247–263
(11) K. Byleen, J. Meakin, F. Pastijn, "The fundamental four-spiral semigroup" *J. of Algebra* , **54** (1978) pp. 6–26
(12) Junghenn (1996) "Operator semigroup compactifications".Transactions of the American Mathematics Sociey. Vol 348, No3. pp 1051 – 1073.

3/4/2009