An Optimizations Problem with First Order Conditions

Dr. Mayank Pawar[1],Vijendra Rawat[1], Dr. Sanjeev Rajan[2] & Rajeev Kumar[3]

[1]Deptt. of Mathematics, College of Engineering,
Teerthankar Mahaveer University, Moradabad. India
[2]H.O.D. Mathematics Hindu college, Moradabad. India
[3]Department of Computer Science, College of Engineering,
Teerthanker Mahaveer University Moradabad. India
Email: Vijendr.rawat@gmail.com

ABSTRACT:-In this paper we consider the optimization problem and define the first order condition that holds the optimization problem. Section 1.1 defines the first order conditions; section 1.2 gives the some example of unconstrained problem.
[Mayank Pawar,Vijendra Rawat, Sanjeev Rajan & Rajeev Kumar. **An Optimizations Problem with First Order Conditions.** Researcher. 2011;3(6):68-71]. (ISSN: 1553-9865). http://www.sciencepub.net.

Keywords: optimization, Local minimum point, global minimum point.

Introduction:- We consider optimization problems of the form

$$\min \quad f(x)$$
$$\text{subject to } x \in \Omega \qquad (1)$$

where f is a real-valued function and Ω., the feasible set, is a subset of E^n.The first order conditions that must hold at a solution point of equation (1). These conditions are simply extensions to E^n of the well-known derivative conditions for a function of a single variable that hold at a maximum or a minimum point.

1.1 FIRST-ORDER CONDITIONS

In an investigation of the general problem (1) we distinguish two kinds of solution points: local minimum points, and global minimum points.

Definition. A point $x^* \in \Omega$ is said to be a relative minimum point or a local minimum point of f over Ω if there is an Q>0 such that $f(x) \geq f(x^*)$ for all $x \in \Omega$ within a distance Q of x^* (that is , such that $f(x) \in \Omega$ and $| x - x^*| < Q$.If $f(x) > f(x^*)$
for all $x \in \Omega$, $x \neq x^*$,within a distance Q of x^*, then x^* is said to be a strict relative minimum point of f over Ω.

Definition. A point $x^* \in \Omega$ is said to be a global minimum point of f over Ω if $f(x) \geq f(x^*)$ for all $x \in \Omega$ .If $f(x) > f(x^*)$ for all $x \in \Omega$, $x \neq x^*$, then x^* is said to be a strict global minimum point of f over Ω. In formulating and attacking problem (1) we are, by definition, explicitly asking for a global minimum point of f over set Ω..Practical reality , however, both from
the theoretical and computational viewpoint ,dictates that we must in many circumstances be content with a relative minimum point .In deriving necessary condition based on the differential calculus ,for instance ,or when searching for the minimum point by convergent stepwise procedure ,comparisons of the values of nearby points is all that is possible and attention focuses on relative minimum points. Global conditions and global solutions can, as a rule ,only be found if the problem possesses certain convexity properties that is essentially guarantee that any relative minimum is global minimum .Thus ,in formulating and attacking problem (1) we shall , by the dictates of practicality ,usually consider ,implicitly ,that we are asking for a relative minimum point .If appropriate conditions hold, this will also be a global minimum point.

Feasible Directions

To derive necessary conditions satisfied by a relative minimum point x^*,the basic idea is to consider movement away from the point in some given direction. Along any given direction the objective function can be regarded as a function of a single variable , the parameter defining movement in this direction ,and hence the ordinary calculus of a single variable is applicable . Thus given $x \in \Omega$ we are motivated to say that a vector d is a feasible direction at x if there is an $\acute{\alpha} > 0$ such that $x + \alpha d \in \Omega$ for all $\alpha, 0 \leq \alpha \leq \acute{\alpha}$. With this simple concept we can state some simple conditions satisfied by relative minimum points.

Proposition 1 (First-order necessary conditions).
Let Ω be a subset of E^n and let $f \in C^1$ be a function on Ω .If x^* is a relative minimum point of f over Ω,then for $d \in E^n$ that is a feasible direction at x^*, we have $\nabla f(x^*)d \geq 0$.

Proof. For any α ,$0 \leq \alpha \leq \acute{\alpha}$., the point $x(\alpha) = x^* + \alpha d \in \Omega$. . For $0 \leq \alpha \leq \acute{\alpha}$ define The function $g(\alpha) = f(x(\alpha))$. Then g has a relative minimum at α= 0.A typical g is shown In Fig 1.1 . By ordinary calculus we have

$$g(\alpha) - g(0) = g'(0)\alpha + o(\alpha), \qquad (2)$$

where o(α) denotes terms that go to zero faster then α. . If g'(0) < 0 then ,for sufficiently small values of α > 0 , the right side of (2) will be negative ,and hence g(α) - g(0) < 0 , which contradicts the minimum nature of g(0).

Thus g'(0) = ∇ f(x*)d ≥ 0.

A very important special case is where x* is in the interior of Ω (as would be the case if Ω = E^n). In this case there are feasible directions emanating in every direction from x*, and hence ∇ f(x*)d ≥ 0 for all d ∈ E^n .This implies ∇ f(x*)= 0. We state this important result as corollary.

Corollary. (Unconstrained case). Let Ω be a subset of E^n , and let f ∈ C^1 be a function' On Ω . If x* is a relative minimum point of f over Ω and if x*is an interior point of Ω then ∇f(x*) = 0.

The necessary condition in the pure unconstrained case lead to n equations (one for each component of ∇f) in n unknown (the components of x*),which in many cases can be solved to determine the solution.In practice however , an optimization problem is solved directly without explicitly attempting to solve the equation arising from necessary conditions.. Nevertheless , these conditions from a foundation for the theory.

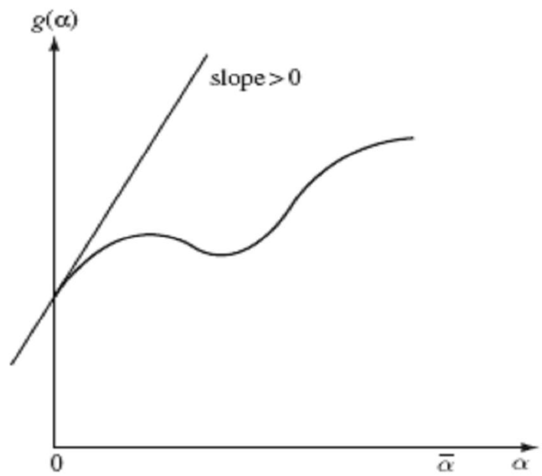

Fig 1.1. Construction for proof

Example 1. Consider the problem

minimize $f(x_1,x_2) = x^2_1 - x_1x_2 + x^2_2 - 3\, x_2.$

There are no constraints , so Ω = E^2 . Setting the partial derivatives of f equal to zero yields the two equations

2x1 - x2 = 0

x1 + 2 x2 = 3 .

These have the unique solution $x_1 = 1$, $x_2 = 2$,. which is a global minimum point of f.

Example 2. Consider the problem

minimize $f(x_1,x_2) = x^2_1 - x_1 + x_2 + x_1 x_2.$

subject to $x_1 \geq 0$, $x_2 \geq 0$

This problem has a global minimum at $x_1 = 1/2$, $x_2 = 0$. At this point

$$\frac{\partial f}{\partial x_1} = 2x_1 - 1 + x_2 = 0$$

$$\frac{\partial f}{\partial x_2} = 1 + x_1 = \frac{3}{2}$$

Thus ,the partial derivatives do not both vanish at the solution, but since any feasible direction must have x_2 component greater than or equal to zero, we have $\nabla f(x^*)d \geq 0$
for all $d \in E^2$ such that d is a feasible direction at point (1/2,0).

1.2 EXAMPLES OF UNCONSTRAINED PROBLEMS

Unconstrained optimization problems occur in a variety of contexts, but most frequently when the problem formulation is simple. More complex formulations often involve explicit functional constraints. However, many problems with constraints are frequently converted to unconstrained problems by using the constraints to establish relations among variables, thereby reducing the effective number of variables. We present a few examples here that should begin to indicate the wide scope to which the theory applies.

Example 1 (Production). A common problem in economic theory is the determination of the best way to combine various inputs in order to produce a certain commodity. There is known production function $f(x_1, x_2 \ldots\ldots, x_n)$ that gives the amount of the commodity produced as a function of the amount x_i of the inputs, i = 1,2,3………,n. The unit price of the produced commodity is q, and the unit prices of the inputs are $p_1, p_2, \ldots\ldots\ldots p_n$.. .The producer wishing to maximize profit must solve the problem

$$\text{maximize} \quad q\, f(x_1, x_2 \ldots\ldots, x_n) - p_1 x_1 - p_2 x_2 \ldots - p_n x_n.$$

The first-order necessary conditions are that the partial derivatives with respect to the x_i's each vanish. This leads directly to the n equations

$$q\frac{\partial f}{\partial x}(x_1, x_2, x_3 \ldots\ldots x_1) \qquad\qquad i = 1, 2, 3 \ldots., n.$$

These equations can be interpreted as stating that, at the solution, the marginal value due to a small increase in the ith input must be equal to the price p_i.

Example 2 (Approximation). A common use of optimization for the purpose of function approximation .Suppose ,for example , that through an experiment the value of a function g is observed at m points , $x_1, x_2 \ldots\ldots, x_m$ .Thus ,values $g(x_1), g(x_2) \ldots\ldots \ldots.., g(x_m)$ are known .We wish to approximate the function by a polynomial.

$$h(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_0$$

of degree n (or less) ,where $n < m$. Corresponding to any choice of the approximating polynomial , there will be a set of errors $\varepsilon_k = g(x_k) - h(x_k)$. We define the best approximation as the polynomial that minimizes the sum of the squares of these errors; that is ,minimizes

$$\sum_{k=1}^{m} (\varepsilon)^2$$

This in turn means that we minimize

$$f(a) = \sum_{k=1}^{m} \left[g(x_k) - (a_n x_k^n + a_{n-1} x_k^{n-1} + \ldots.. + a_0) \right]^2$$

with respect to $a = (a_0, a_1 \ldots\ldots, a_n)$ to find the best coefficients.This is a quadratic expression in the coefficient a. To find a compact representation for this objective we define

$$q_{ij} = \sum_{k=1}^{m} (x_k)^{i+j}, \; b_j = \sum_{k=1}^{m} g(x_k)(x_k)^f \; and \;\; c = \sum_{k=1}^{m} g(x_k)^2$$

Then after a bit of algebra it can be shown that

$$f(a) = a^T Q a - 2 b^T a + c$$

where $Q = [q_{ij}]$, $b = ((b_1, b_2 \ldots\ldots, b_{n+1})$
The first–order necessary conditions state that gradient of f must vanish. This leads directly to the system of n+1 equations

$$Qa = b$$

.This can be solved to determine a.

Conclusion :-In this paper we consider the optimization problems in equation (1).In equation (1) function is real-valued and Ω., the feasible set, is a subset of E^n.The first order conditions that must hold at a solution point of equation (1). These conditions are simply extensions to E^n of the well-known derivative conditions for a function of a single variable that hold at a maximum or a minimum point.

References:-

(1)Hadley, G., Nonlinear and Dynamic Programming, Addison-Wesley, Reading, Mass., 1964.
(2) Fiacco, A. V., and McCormick, G. P., Nonlinear Programming: Sequential Unconstrained Minimization Techniques, John Wiley, New York, 1968
(3) Zangwill, W. I., Nonlinear Programming: A Unified Approach, Prentice-Hall, Englewood Cliffs, N.J., 1969
(4) Luenberger, D. G., Optimization by Vector Space Methods, John Wiley, New York, 1969.

6/4/2011