Boundary Curvature Of The Numerical Range Of 2 × 2 Matrices

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ABSTRACT: For any $n \times n$ matrix A, define the constant $M(A)$ by $sup\, \frac{dist(\lambda,\sigma(A))}{r_\lambda(A)}$. Here λ is a point on the smooth portion of the boundary $\partial W(A)$ of the numerical range of A, and $r_\lambda(A)$ is radius of curvature of $\partial W(A)$ at this point, and dist$(\lambda, \sigma(A))$ is the distance from λ to the spectrum of A. Where the supremum is taken over all a points λ in $\partial W(A)$ with nonzero curvature $r_\lambda(A)$. Since $r_\lambda(A) = 0$ implies that λ is an eigenvalue of A and hence dist$\big(\lambda, \sigma(A)\big) = 0$, we have dist$\big(\lambda, \sigma(A)\big) \leq M(A) r_\lambda(A)$ for all λ in $\partial W(A)$. In this paper we compute the $M(A)$ for 2×2 matrix A . 200 Mathematics Subject Classification. Primary 47A12; Secondary 15A42, 14H50.
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1. INTRODUCTION

Let A be an $n \times n$ matrix with complex entries: $\mathbb{C}^{n\times n}$. The numerical range of A is defined as

Where $<.,.>$ and $||.||$ are the standard scalar product and norm on $\mathbb{C}^n$, respectively.
Thus the numerical range of an operator, like spectrum, is a subset of the complex plane whose geometrical properties should say something about the operator.
We consider the quantity $M_n = \sup\{M(A){:}\, A\, , n - by - n\; matrix\}$.
In [1] we have that $\frac{n}{2}\sin\left(\frac{\pi}{n}\right) \leq M_n \leq \frac{n}{2}$ for $n \geq 2$.
If $n = 2$, then the upper and lower bounds coincide, so that $M_n = 1$. In fact, it can be seen from problem [1] that $M(A) \leq 1$ for any 2×2 matrix A, and $M(A) = 1$ when A has equal eigenvalues or, equivalently, when $W(A)$ is a circular disc.

2. PRELIMINARIES AND AUXILIARY RESULTS

Let A be an $n \times n$ matrix with complex entries: $\mathbb{C}^{n\times n}$. The numerical range of A is defined as

Where $<.,.>$ and $||.||$ are the standard scalar product and norm on $\mathbb{C}^n$, respectively.

2.1. Proposition. For an operator A on a Hilbert space H:
(a) $W(A)$ is invariant under unitary similarity,
(b) $W(A)$ lies in the closed disc of radius $||A||$ centered at the origin,
(c) $W(A)$ contains all the eigenvalues of A.
(d) $W(A^*) = \{\bar{\lambda}{:}\, \lambda \epsilon W(A)\}$,
(e) $W(I) = \{1\}$. More generally, if α and β are complex numbers, and A a bounded operator on H, then $W(\alpha A + \beta I) = \alpha W(A) + \beta$.
(f) If H is finite dimensional then $W(A)$ is compact.
Proof. In [1]. ■

One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toepliz-Hausdorff Theorem. Other important property of $W(A)$ is that its closure contains the spectrum of the operator.

$W(A)$ is a connected set with a piecewise analytic boundary $\partial W(A)$ for details see [2].
Hence, for all but finitely many point $\lambda \in \partial W(A)$, the radius of curvature $r_\lambda(A)$ of $\partial W(A)$ at λ is well defined. By convention, $r_\lambda(A) = 0$ if λ is a corner point of $W(A)$, and $r_\lambda(A) = \infty$ if λ lies inside a flat portion of $\partial W(A)$. Let $\mathrm{dist}\big(\lambda, \sigma(A)\big)$ denote the distance from λ to $\sigma(A)$, we define $M(A)$ the smallest constant such that

$$\mathrm{dist}\big(\lambda, \sigma(A)\big) \leq M(A) r_\lambda(A), \qquad (1)$$

for all $\lambda \epsilon \partial W(A)$, where $r_\lambda(A)$ is defined.
By Donoghes theorem $\mathrm{dist}\big(\lambda, \sigma(A)\big) = 0$ whenever $r_\lambda(A) = 0$. Therefor, $M(A) = 0$ for all convexoid element A. Recall that convexoid element is an element such that its numerical range coincides with the convex hull of its spectrum. For non-convexoid A,

$$sup \frac{dist(\lambda, \sigma(A))}{r_\lambda(A)}, \qquad (2)$$

where the supremum in the right-hand side is taken along all points $\lambda \epsilon \partial W(A)$ with finite non-zero curvature. Computation of $M(A)$ for arbitrary A is an interesting problem. Computation of $M(A)$ for arbitrary $n \times n$ matrix A is also an interesting open problem.

For we $n > 3$, do not have an exact value of

$$M_n = \sup\{M(A): A \epsilon \mathbb{C}^{n\times n}\}, (3)$$

the question whether there exists a universal constant $M = sup_n M_n$, posed by Mathias [4].
In [1] the authors have proved that

$$\frac{n}{2}\sin\left(\frac{\pi}{n}\right) \leq M_n \leq \frac{n}{2}. (4)$$

In [5] the author find a sequence of $n \times n$ Toeplitz nilpotent matrices A_n with $M(A_n)$ algorowing asymptotically as $\log n$.
Hence, the ansewer to Mathias question is negative.
However, the lower bound in (4) is still of some interest, at least for small values of
The question of the exact rate of growth of M_n (is it $\log n$, or n, or something in between) remains open.

3. THE NUMERICAL RANGE OF A TWO BY TWO MATRIX
In this section we prove that the numerical range of a two by two matrix(i.e. an operator on a two dimensional Hilbert space) assumes one of the following three forms:
(a) A single point, if the operator is a scalar multiple of the identity,
(b) a line segment joining the eigenvalues, if the operator is normal with two distinct eigenvalues, or
(c) an elliptical disc with foci at the eigenvalues, if the operator has distinct eigenvalues, but is not normal.

3.1. Propositon. The numerical range of a two by two cross-diagonal matrix is either an elliptical disc with foci at the eigenvalues, or a line segment joining the eigenvalues.
Proof. We have A= $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ where a and b are complex numbers. First suppose a and b are positive. We may assume $0 < b \leq a$, else take adjoints and use Propositon 2.1(d). We parameterize the unit(column) vectors of $\mathbb{C}^2$ as follows: $x = x(\theta, \varphi, t) = e^{i\varphi}\big[t, e^{i\theta}\sqrt{1-t^2}\big]^T$, then employing the parameterization and doing some computation:

which (because $max_{0\leq t\leq 1} t\sqrt{1-t^2} = \frac{1}{2}$) describes either:
* The line segment $[-a, a]$ if $a = b$ (in which case $\pm a$ are the eigenvalues of A), or
* The ellipse with center at the origin, horizontal major axis of length $a + b$ and vertical minor axis of length $a - b$

if $a \neq b$.
Now from analytic geometry we know that for an ellipse with foci $\pm F$ on the real axis, major semi-axis of length M and minor semi-axis of length m, we have $F^2 + m^2 = M^2$. In our case $M = \frac{(a+b)}{2}$ and $m = \frac{(a-b)}{2}$, so $F = \pm\sqrt{M^2 - m^2} = \pm\sqrt{ab}$, i.e. the foci of $W(A)$ are the eigenvalues of A. ■

To summarize our work to this point:
For any non- negative numbers a and b the numerical range of the matrix $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ is a (possibly degenerate) elliptical disc with foci at the eigenvalues.
Precisely the same result holds if a and b are arbitrary complex numbers. Indeed, write both in polar form: $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$, and observe that if $S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\alpha-\beta}{2}} \end{bmatrix}$ then $SAS^{-1} = e^{i\frac{\alpha+\beta}{2}}\begin{bmatrix} 0 & |a| \\ |b| & 0 \end{bmatrix}$.
From Proposition 2.1 (e) and the result just proved for non-negative a and b we see that $W(A)$ is an ellipse with foci at $\pm\sqrt{|a||b|}e^{i\frac{\alpha+\beta}{2}} = \pm\sqrt{ab}$ = the eigenvalues of A.
Taking into account the lengths of the axes of our ellipse, we can summarize the work to this point as follows:
3.2. Proposition. If A= $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ where $a, b \in \mathbb{C}$, then $W(A)$ is the (possibly degenerate) ellipse with center at the origin, whose foci are the eigenvalues $\pm\sqrt{ab}$ of A, whose major axis has length $|a| + |b|$ and whose minor axis has length $\big||a| - |b|\big|$ ■

3.3.The Ellipse Theorem. If A is a linear transformation on $\mathbb{C}^2$, then $W(A)$ is a (possibly degenerate) elliptical disc.
Proof. It is enough to consider A with trace zero (else replace A by $A -$ (trace $\frac{A}{2})I$, and use the transformation law (c) of Proposition 2.1). In view of what we've done with cross diagonal matrices, the Ellipse Theorem will follow immediately from the following result, which is interesting in its own right:
If A is a two by two complex matrix with trace zero then A is unitarily equivalent to a matrix with zero-diagonal. continuo proof in[1]. ■

4. COMPUTE THE $M(A)$ FOR 2×2 MATRIX A

Theorem 4.1. Let A be a 2×2 matrix with the eigenvalues λ_1, λ_2, and let $s = (trace(A^*A) - |\lambda_1|^2 - |\lambda_2|^2)^{\frac{1}{2}}$. Then $M(A) = 0$ if $s = 0$ and

$$M(A) = \frac{s}{\sqrt{s^2 + |\lambda_1 - \lambda_2|^2}}, \quad (5)$$

otherwise.

Proof. The matrix A is normal if and only if $s = 0$; in this case $M(A) = 0$. For $s > 0$, the matrix A is unitarily irreducible, and $W(A)$ is an ellipse with minor axis $2b = s$ and major axis $2a = \sqrt{s^2 + |\lambda_1 - \lambda_2|^2}$. The foci are, of course, located at the eigenvalues.
For a current point $P \in \partial W(A)$, if $\partial W(A) = acos\theta + ibsin\theta$. We have

$$\begin{cases} x = acos\theta \Longrightarrow x' = -asin\theta \Longrightarrow x'' = -acos\theta. \\ y = bsin\theta \Longrightarrow y' = bcos\theta \Longrightarrow y'' = -bsin\theta. \end{cases} \quad (6)$$

Then

$$k_P(A) = \frac{|x'y'' -}{(x'^2 +}$$

$$r_P(A) = \frac{(a^2sin^2\theta + b^2cos^2\theta)^{\frac{3}{2}}}{|ab|} \quad (7)$$

Let x denot the distance from P to the closest eigenvalue. Then

From (7) and (8) the radius of curvature at the point P is

$$r_P(A) = \frac{(x(2a-x))^{\frac{3}{2}}}{|ab|}. \quad (9)$$

So that $M(A) = max\{f(x): a-c \leq x \leq a\}, where \ \ f(x) = \frac{abx}{x^{\frac{1}{2}}(2a-x)^{\frac{3}{2}}}$

Elementary calculus shows that:

And $\frac{b}{a} = \frac{s}{\sqrt{s^2+|\lambda_1-\lambda_2|^2}}$ So that

$$M(A) = \frac{s}{\sqrt{s^2+|\lambda_1-\lambda_2|^2}}. \ \blacksquare$$

4. MAIN RESULTS

Result 1.

In this paper we show that the numerical range of a two by two matrix(i.e. an operator on a two dimensional Hilbert space) assumes one of the following three forms:

(a) A single point, if the operator is a scalar multiple of the identity,

(b) a line segment joining the eigenvalues, if the operator is normal with two distinct eigenvalues, or

(c) an elliptical disc with foci at the eigenvalues, if the operator has distinct eigenvalues, but is not normal.

In other words:

The numerical range of an operator on a two dimensional Hilbert space is a (possibly degenerate) elliptical disc with foci at the eigenvalues.

Result 2.

We consider the quantity $M_n = \sup\{M(A): A\ , n-by-n\ matrix\}$.

In [1] we have that $\frac{n}{2}\sin\left(\frac{\pi}{n}\right) \leq M_n \leq \frac{n}{2}$ for $n \geq 2$.

If $n = 2$, then the upper and lower bounds coincide, so that $M_n = 1$. In fact, it can be seen from problem [1] that $M(A) \leq 1$ for any 2×2 matrix A, and $M(A) = 1$ when A has equal eigenvalues or, equivalently, when $W(A)$ is a circular disc. Also compute $M(A)$ for 2×2 matrix A is

where λ_1, λ_2, eigenvalues matrix A, and $s = (trace(A^*A) - |\lambda_1|^2 - |\lambda_2|^2)^{\frac{1}{2}}$. $\blacksquare$

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