Skew version of rings

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Abstract: The paper deals with Armendariz rings, their relationships with well known rings. Then we treat generalizations of Armendariz rings, such as McCoy ring, abelian ring and their links. In this paper, we consider a skew version of some classes of rings, with respect to a ring endomorphism ***α***.

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1. Introduction

The theory of rings is one of the important parts of algebra dealing with structures required to develop other algebraic theories and their applications. Plenty of generalizations of the classical rings like the ring of integers; associative rings have been introduced and successfully have been implemented to various branches of science.

The paper aims to investigate a class of rings called *Armendariz* rings, which generalizes fields and integral domains. These rings are associative with identity and they were introduced by Rege and Chhawchharia (1997). Throughout this research all rings are associative with identity unless otherwise stated. A ring R is called *Armendariz* if whenever the product of any two polynomials in $R[x]$ over R is zero, then so is the product of any pair of coefficients from the two polynomials. In 1974 E.P.Armendariz proved that if the product of two polynomials whose coefficients belong to a ring without nonzero nilpotent elements equals zero, then all possible pair wise products of coefficients of these polynomials equal zero.

Recall that for a ring R with a ring endomorphism, $\alpha : R \rightarrow R$ and an α-derivation δ of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all a,b in R. The ore extension $R[x; \alpha, \delta]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x + \delta(r)$ for any r in R. If $\delta = 0$, we write $R[x; \alpha]$ and is called a skew polynomial ring (also Ore extension of endomorphism type.)

Some properties of skew polynomial rings have been studied in Hong et al (2003) and Hong et al (2006). According to Krempa (1996), an endomorphism α of a ring R is called *rigid*, if

$a\alpha(a)=0$ implies $a=0$ for a in R. In Hong et al (2000), a ring R is called *α-rigid* if there exist a rigid endomorphism α of R. Any rigid endomorphism of a ring is a monomorphism and *α-rigid* rings are reduced (Hong et al, 2000). Hong et al (2006), introduced *α-Armendariz* rings, which are a generalization of α –rigid rings and Armendariz rings. R is called *α-Armendariz* ring, if whenever the product of any two polynomials in $R[x; \alpha]$ over R is zero, then so is the product of any pair of coefficients from the two polynomials.

The organization of the paper is as follows. First, we consider the relationship between Armendariz rings and some other classes of rings (section 1), then we treat generalizations of Armendariz rings (section 2), then we consider the skew version of some clesses of rings (section 3).

In this paper, α is an endomorphism of R.

2. Armendariz rings and other rings

In this section we explore relationships between several classes of rings, provide examples confirming these relationships and prove some statements about them.

Recall that a ring R is said to be *von Neumann regular*, if a in aRa for any element a of R. Every Boolean ring is von Neumann regular. Reduced rings are Armendariz, but the converse does not hold. Anderson and Camillo (1998) proved that a von Neumann regular ring is Armendariz, if and only if it is reduced.

By Kaplansky (1968), a ring R is called a *right p.p-ring*, if the right annihilator of each element of R is generated by an idempotent. A ring R is called *Baer*, if the right annihilator of every nonempty subset of R is generated by an idempotent. Clearly Baer ring is right *p.p-ring*. Any Baer ring has nonzero central nilpotent element and so a commutative Baer ring is Armendariz. Reduced rings can be included in the class of Armendariz rings and the class of semicommutative rings, and both of them are abelian.

So it is natural to explore the relationships between them.

A ring is said to be *semicommutative*, if it satisfies the following condition: whenever elements *a,b* in *R* satisfy $ab=0$, then $aRb=0$.

Semicommutative rings are abelian, but the converse does not hold. Kim et al (2000) showed that abelian ring need not be semicommutative.

Another class of rings is *Guassian, which considered by Anderson and Camillo (1998).* The content $c(f)$ of $f(x)$ is the ideal of R generated by the coefficients of $f(x)$. A commutative ring R with identity is *Guassian,* if $c(fg)=c(f)\,c(g)$ for all $f(x)$, $g(x)$ in $R[x]$. Guassian rings are Armendariz, but the converse is false. Any integral domain is Armendariz, but it is not necessarily Guassian. A field is Guassian, hence it is Armendariz.

Recall that, a ring R is called *symmetric*, if $abc=0$ implies $acb=0$ for *a, b* and *c* in *R.* A ring *R* *reversible* provided $ab=0$ implies $Rba=0$ for *a, b* in *R*. Semicommutative ring is a generalization of reversible ring. A ring is said to be *semicommutative,* if it satisfies the following condition whenever elements *a, b* in *R* satisfy $ab=0$, then $aRb=0$. The following implications hold by simple computation: (pourtaherian and Rakhimov, 2009).

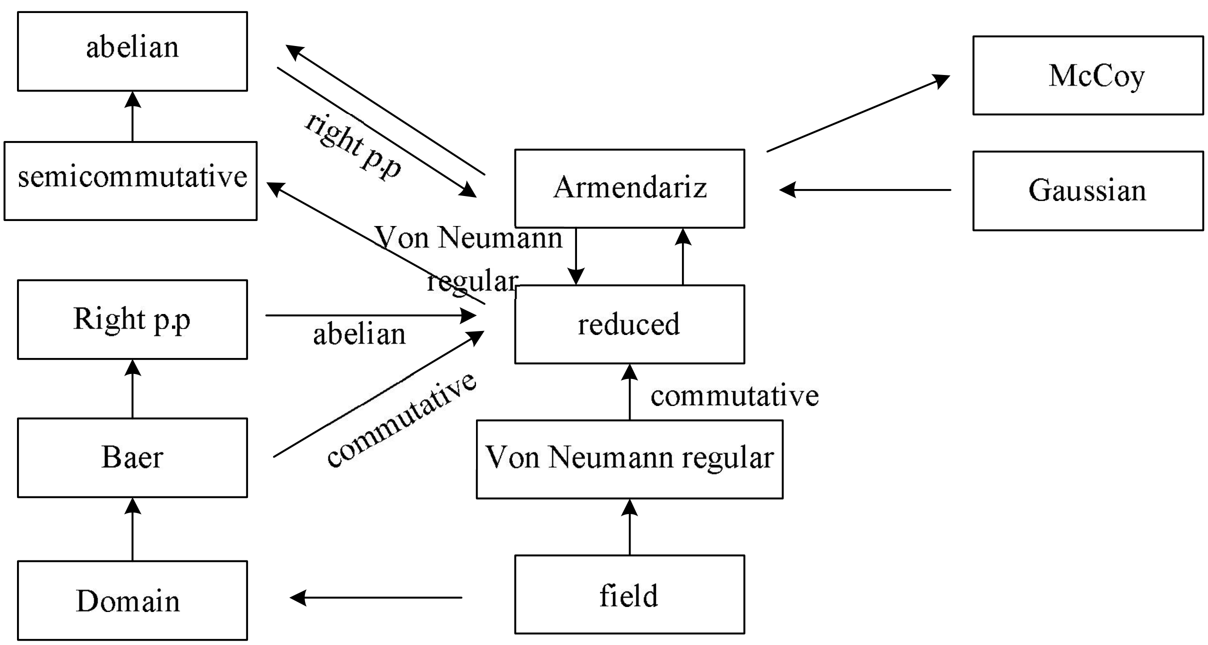
Reduced→symmetric→reversible→semicom mutatie → abelian.

3. Generalizations of *Amendariz* rings

Abelian rings are a generalization of Armendariz rings. Kim and Lee (2000), proved that Armendariz rings are *abelian* (i.e. every its idempotent is central), McCoy rings are another generalization of Armendariz rings, recall that a ring is a *left McCoy*, if whenever $g(x)$ is a right zero-divisor in $R[x]$ there exists a non-zero element c in R such that $cg(x)=0$. *Right McCoy* ring is defined dually. A ring is said to be *McCoy ring*, if it is both left and right *McCoy.*

Armendariz rings are McCoy (Rege et al, 1997), the converse does not hold. Commutative rings are McCoy (Scott, 1954), but there are examples of commutative non-Armendariz rings.

We show previous results about the connection among Armendariz and other rings with the following diagram.



4. Skew version of rings

In this section, we consider a skew version of some classes of rings, with respect to a ring endomorphism α.

When α is the identity endomorphism, this coincides with the notion of ring.

Kwak(2007), called an endomorphism α of a ring *R*, right (left) *symmetirc* if whenever $abc=0$ implies

$ac\alpha(b)=0$.($\alpha(b)ac=0$) for *a,b,c* in *R* . A ring *R* is called right (left) α *-symmetric* if there exists a right (left) symmetric endomorphism α of R. R is α -*symmetric* if it is right and left α -*symmetric*.

Obviously, domains are α -symmetric for any endomorphism α.

Başer et al(2009), called a ring R right (left) α -*reversible* if whenever $ab=0$ for a,b in R then $b\,\alpha(a)=0$ ($\alpha(b)\,a=0$). R is called α – *reversible* if it is both right and left α - *reversible*.

α -symmetric rings are α –reversible.

Başer et al(2008), defined the notion of an α -semicommutative ring with the endomorphism α as ageneralization of α -rigid ring and an extension of semicommutative ring.

An endomorphism α of a ring R is called *semicommutative* if $ab=0$ implies $aR\,\alpha(b)=0$ for a, b in R. A ring R is called α -*semicommutative* if there exists a semicommutative endomorphism α of R.

Domain is both semicommutative and α -semicommutative for any endomorphism α.
The next example shows that there exists an α -semicommutative ring which is not domain.

Example. Let $R=\left\{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a, b \text{ in } \mathbb{Z}\right\}$ and α be an endomorphism of R defined by $\alpha\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}=\begin{pmatrix} a & 0 \\ -b & a \end{pmatrix}$.
R is α –semicommutative ring.

Proposition. α -semicommutative ring R with $\alpha(1)=1$ is abelian, where 1 is the identity of R.

The following example shows that the condition $\alpha(1)=1$ can't be dropped.

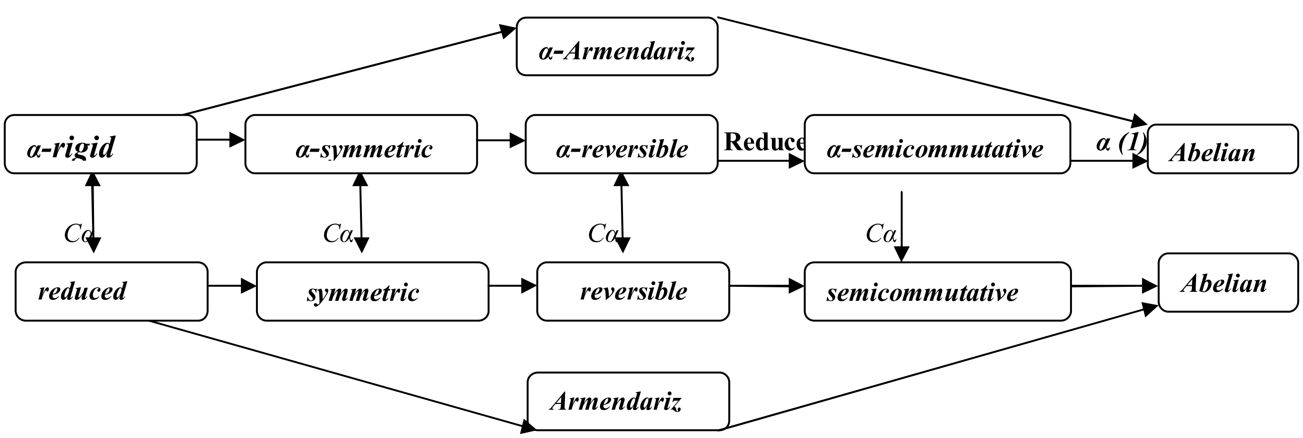
Example. Let $R=\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \text{ in } \mathbb{Z}\right\}$ and α be an endomorphism of R defined by $\alpha\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$ ($\alpha(1)\neq 1$). R is α –semicommutative ring. Two idempotents of R (i.e., $\begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$) aren't central, so R is not abelian.

α -symmetric rings are α -semicommutative.

Ben Yakoub and Louzari(2009), called a ring R satisfies the condition ($C\alpha$) if whenever $a\alpha(b)=0$ with a, b in R, then $ab=0$.
Clearly, α -compatible ring satisfies the condition ($C\alpha$).
They proved that, for a ring R with ($C\alpha$) condition, R is reversible (resp. symmetric) if and only if R is α –reversible (resp. α –symmetric).

Proposition. For a ring R with condition ($C\alpha$). If R is α –semicommutative then it is semicommutative ring.

We show previous results about the connection among skew version of rings with the following diagram.



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