Solving system of two-dimensional nonlinear Volterra-Fredholm integro-differential equations by He's Variational iteration method

Alireza gholami

Department of mathematics, Payame Noor University, Tehran, Iran
alirezagholami244@yahoo.com

Abstract: In this paper, He's variational intration method is imployed successfully for solving systems of the partial mixed Volterra-Fredholm integro-differential equations, The variational interation method (VIM) formula is derived and the lagrange multiplier can be effectively identified. Moreover, this technique does not require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computations. The results seems to show that the method is very effective and convenient for solving such equations.
[Alireza gholami. **Solving system of two-dimensional nonlinear Volterra-Fredholm integro-differential equations by He's Variational iteration method.** *Researcher* 2015;7(6):81-85]. (ISSN: 1553-9865). http://www.sciencepub.net/researcher. 14

Keywords: Nonlinear mixed Volterra-Fredholm integral equations, Variational iteration method.
MSC2010: 35R09

1 Introduction

The partial integral equations and integro-differential equatioans arise in many applied problems of mechanics, physics, engineering, and even biology, there is a few systematic treatise of threir theory, methods, and applications so far. Recently, many different basic functions have been used to estimate the solution of integral equations, such as orthogonal functions and wavelets. Haar wavelets are the simplest orthogonal wavelet with compact support, and they have been used in different numerical approximation problems[3,4].

We consider a systems of the partial mixed Volterra-Fredholm integro-differential equations of the form.

$$a(s,t)D_s u(s,t) + b(s,t)D_t u(s,t) + c(s,t)u(s,t) = f(s,t)$$

$$+\lambda \int_0^t \int_\Omega K\big(s,t,x,y,u(x,y)\big)dxty \quad (s,t) \in W = (O,T) \times \Omega), \qquad (1.1)$$

Where D_x and D_t are the ordinary differential operators, with $m_{ij} \in Z^+$ respectively of the forms

$$D_s = \begin{pmatrix} D_s^{m_{11}} & D_s^{m_{12}} & \cdots & D_s^{m_{1n}} \\ D_s^{m_{21}} & D_s^{m_{22}} & \cdots & D_s^{m_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ D_s^{m_{n1}} & D_s^{m_{n2}} & \cdots & D_s^{m_{nn}} \end{pmatrix} \quad D_t = \begin{pmatrix} D_t^{m_{11}} & D_t^{m_{12}} & \cdots & D_t^{m_{1n}} \\ D_t^{m_{21}} & D_t^{m_{22}} & \cdots & D_t^{m_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ D_t^{m_{n1}} & D_t^{m_{n2}} & \cdots & D_t^{m_{nn}} \end{pmatrix},$$

$$f(s,t) = (f_1(s,t), f_2(s,t), \cdots, f_n(s,t))^t$$

$$u(s,t) = (u_1(s,t), u_2(s,t), \cdots, u_n(s,t))^t \qquad (1.2)$$

$$k(s,t,x,y,f(x,y)) = (k_1(s,t,x,y,f(x,y)), k_2(s,t,x,y,f(x,y)), \cdots, k_n(s,t,x,y,f(x,y)))^t$$

With given supplementary initial conditions, $a(s,t), b(s,t)$ and $c(s,t)$ are given continuous functions, where $u(s,t)$ is an unknown function which should be determined, the functions $k(s,t,x,y,u)$ and $f(s,t)$ are analytical functions on $W = [O,T] \times \Omega$ and $\times \Omega^2$, respectively. X is a (closed) bounded region in $R^n (n = 1,2,3)$ with (piecewise) smooth boundary $\partial\Omega$ [1].

Since solving partial mixed Volterra-Fredholm integral equations, especially system of the partial mixed Volterra-Fredholm integral equation are new subject in physical and mathematical problems, there are only a few techniques for solving these types of equations.

Recently a He's homotopy perturbation method has been developed for solving systems of the mixed Volterra-Fredholm integral equations [5], Babolian [9] used He's homotopy perturbation method for solving a nonlinear system of two dimensional Volterra-Fredholm integral equations, a calss of two-dimensional nonlinear Volterra integral equations solved by using Legendre polynomials [8], Maleknejad et al, have applied two dimensional Block-Pluse functions (2D-BPFs) for solving nonlinear mixed Volterra-Fredholm integral equations [6], Yousefi et al, solving nonlinear mixed Volterra-Fredholm integral equations by He's variational iteration method[7], and Babolian et al, have applied two dimensional triangular functions for solving nonlinear Volterra-Fredholm integral equations [2].

One the other hand, there are many numerical methods for solving system of integral equations, but in mixed Volterra-Fredholm integral equation cases, a few works have been done [5,9].

The variational itreration method has proved to be one of the useful techniques in solving numerous linear and nonlinear differential equations. variational itreration method was fisrt proposed by He [10] and was successfully applied to autonomous ordinary differential equations [11], functional integral equatin [12], nonlinear systems of partial differential equations [13], Optimal control based [14], vibration of conservative oscillators [15], Hamilton-Jacobi-Bellman equations [16].

To illustrate the basic concepts of the VIM, we consider the following differential equations:

2 Basic ideas of variational itreration method

$$L\left(u_j(t)\right) + N\left(u_j(t)\right) = g_j(t), \quad j = 1,2,\dots,n. \qquad (2.3)$$

Where L is a linear operator, N is a nonlinear operator and $g(x)$ is a givne continuous function.

$$u_{j,n+1}(t) = u_{j,n}(t) + \int_0^t \lambda(L(u_{j,n}(s)) + N(\tilde{u_j}(s)) - g_j(s))ds, \quad j = 1,2,\dots,n.$$

(2.4)

The basic character of the method is to construct a correction functional for the system, where λ is a general Lagrange multiplier which can be identified optimally via variational theory, u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, ie. $\delta\tilde{u} = O$.

3 Applying the method

In this section, we condider the nonlinear systems of the partial mixed Volterra-Fredholm integro-differential equations which can be written in the form:

$$\frac{\partial u_j(s,t)}{\partial t} + \frac{s^2}{3}\frac{\partial^2 u_j(s,t)}{\partial s^2} = f_j(s,t) + \int_0^t \int_\Omega K_j\big(s,t,x,y,u(x,y)\big)dxdy \quad (s,t) \in W = ([O,T) \times \Omega),$$

$$j = 1,2,\cdots,n. \qquad (3.5)$$

With the initial conditions

$$u_j(s,O) = s, \quad j = 1,2,\cdots,n. \qquad (3.6)$$

For Eq.(3.5) first we take the partial derivative with respect to t_j, $(j = 1,2,\cdots,n)$. We have

$$\frac{\partial^2 u_j(s,t)}{\partial t^2} - \frac{s^2}{3}\frac{\partial^3 u_j(s,t)}{\partial s^2 \partial t} - \frac{\partial f_j}{\partial t} - \int_0^1 K_j\big(s,t,x,t,u(x,t)\big)dx - \int_0^t \int_0^1 \frac{\partial K_j}{\partial t} dxdy = 0.$$

Consider

$$-\int_0^1 K_j\big(s,t,x,t,u(x,t)\big)dx - \int_0^t \int_0^1 \frac{\partial K_j}{\partial t} dxdy, \qquad j = 1,2,\cdots,n.$$

As a restricted variation, we use the variational iteration method in direction t.

Then we have the following iteration sequence:

$$u_{j,n+1}(s,t) = u_{j,n}(s,t) + \int_0^t \lambda[\frac{\partial^2 u_{j,n}(s,\tau)}{\partial \tau^2} - \frac{s^2}{3}\frac{\partial^3 u_{j,n}(s,\tau)}{\partial s^2 \partial \tau} - \frac{\partial f_j}{\partial \tau}(s,\tau)$$
$$- \int_0^1 K_j\left(s,\tau,x,\tau,u_{j,n}(x,\tau)\right)dx - \int_0^\tau \int_0^1 \frac{\partial K_j}{\partial t} dxdy]d\tau.$$

Taking the variation with respect to the independent variable u_n and noticing that $\delta u_{j,n}(0) = 0$, we get

$$\delta u_{j,n+1}(s,t) = \delta u_{j,n}(s,t) + \lambda \delta u'_{j,n}(s,\tau) \Big|_{\tau=t} - \delta\left(\frac{\partial \lambda}{\partial s} u_{j,n}(s,\tau) \Big|_0^t - \int_0^t \frac{\partial^2 \lambda}{\partial s^2} u_{j,n}(s,\tau)d\tau\right)$$

$$= \delta u_{j,n}\left(1 - \frac{\partial \lambda}{\partial s}\right) + \lambda_j \delta u'_{j,n}(s) + \int_0^t \frac{\partial^2}{\partial \tau^2}\delta u_{j,n}(s,t)d\tau \qquad (3.7)$$

Then, the stationary conditions are obtained

$$\frac{\partial^2 \lambda_j(s,\tau)}{\partial s^2} = 0, \quad 1 - \frac{\partial \lambda_j(s,\tau)}{\partial s}\Big|_{\tau=t} = 0, \quad \lambda_j(s,t)\Big|_{\tau=t} = 0, \quad j = 1,2,\cdots,n. \qquad (3.8)$$

Hence, the lagrange multiplier is

$\lambda_j = \tau - t, \quad j = 1,2,\cdots,n.$ (3.9)

And the following iteration formula can be obtained as

$$u_{j,n+1}(s,t) = u_{j,n}(s,t) + \int_0^t (\tau - t)[\frac{\partial^2 u_{j,n}(s,\tau)}{\partial \tau^2} - \frac{s^2}{3}\frac{\partial^3 u_{j,n}(s,\tau)}{\partial s^2 \partial \tau} - \frac{\partial f_j}{\partial \tau}(s,\tau)$$
$$- \int_0^1 K_j\left(s,\tau,x,\tau,u_{j,n}(x,\tau)\right)dx - \int_0^\tau \int_0^1 \frac{\partial K_j}{\partial t} dxdy]d\tau. \quad (3.11)$$

Starting with an initial approximation

$$u_{j,0}(s,t) = s\,, \quad j = 1,2,\cdots,n. \quad (3.12)$$

And using the iteration formula (3.10), we can obtain the successive approximations.

4 Example

In this section, we give some examples to clarify the accuracy of the presented method

Example 1. *Consider the system of integro-differential equations*

$$\frac{\partial u_1(s,t)}{\partial t} - su_2(s,t) - \int_0^t \int_0^1 x^2 u_1^2(x,y)u_2^2(x,y)dxdy = -\frac{1}{15}t^3 + se^t - ste^{-t}$$

(4.12)

$$\frac{\partial u_1(s,t)}{\partial t} - \frac{s^2}{2}\frac{\partial u^2(s,t)}{\partial t} - \int_0^t \int_0^1 y^2 u_1(x,y)u_2(x,y)dxdy = -\frac{1}{8}t^4 + se^t - \frac{1}{2}s^2(e^{-t} - te^{-t})$$

For $s,t \in [0,1]$ and with supplementary conditions

$$u_1(s,0) = s, \quad u_2(s,0) = 0, \quad t \in [0,1]$$

Which the exact solutions are $u_1(s,t) = se^t$and $u_2(s,t) = te^{-t}$. Using the present method we have

$$\frac{\partial^2 u_1(s,t)}{\partial t^2} - s\frac{\partial u_2(s,t)}{\partial t} - se^t + se^{-t} - ste^{-t} + \frac{3}{15}t^2 - \int_0^1 x^2 u_1^2(x,t)u_2^2(x,t)dx$$
$$\frac{\partial^2 u_1(s,t)}{\partial t^2} - \frac{s^2}{2}\frac{\partial^2 u_2(s,t)}{\partial t^2} + \frac{1}{2}t^3 - se^t - s^2e^{-t} + \frac{1}{2}s^2\, te^{-t} - \int_0^1 t^2 u_1(x,t)u_2(x,t)dx$$

We applied the method presented in this paper and solved Eq.(4.12). Hence, the Lagrange multiplier, therefore, can be readily identified

$$\lambda_j = \tau - t, \quad j = 1,2, \quad (4.14)$$

And we obtain the following iteration formula

$$u_{1,n+1}(s,t) = u_{1,n}(s,t) + \int_0^t (\tau - t)[\frac{\partial^2 u_{1,n}(s,\tau)}{\partial \tau^2} - s\frac{\partial^2 u_{2,n}(s,\tau)}{\partial \tau^2}$$
$$-se^\tau + se^{-\tau} - ste^{-\tau} + \frac{3}{15}\tau^2$$
$$- \int_0^1 x^2 u_{1,n}^2(x,\tau)u_{2,n}^2(x,\tau)dx]d\tau \quad (4.15)$$
$$u_{2,n+1}(s,t) = u_{2,n}(s,t) + \int_0^t (\tau - t)[\frac{\partial^2 u_{1,n}(s,\tau)}{\partial \tau^2} - \frac{s^2}{2}\frac{\partial^2 u_{2,n}(s,\tau)}{\partial \tau^2}$$
$$+\frac{\tau^3}{2} - se^\tau - s^2e^{-\tau} + \frac{1}{2}s^2\tau e^{-\tau}$$
$$- \int_0^1 \tau^2 u_{1,n}(x,\tau)u_{2,n}(x,\tau)dx]d\tau \quad (4.16)$$

We start with an initial approximation $u_{1,2}(s,0) = s,\ u_{2,0}(s,0) = 0$ and using the iteration formula (4.15) and (4.16). we get u_n for *n=6,8,10*, and the error function $|u(s,t) - u_n(s,t)|$ and table 1 shows the numerical results obtaines by this approximation.

Example 2. *Consider the following nonlinear system of two-dimensional Volterra-fredholm integro-differential equations*

$$u_1(s,t) - \frac{s}{2}\frac{\partial u_2(s,t)}{\partial t} + \int_0^t \int_0^1 sy(u_1^2(x,y) + u_2(x,y)dxdy = s + t - st + \frac{1}{3}st^2 + \frac{1}{3}st^3$$
$$\frac{\partial u_1(s,t)}{\partial t} + \frac{s^2}{2}u_2(s,t) + \int_0^t \int_0^1 s(u_1(x,y) - u_2^2(x,y)dxdy = 1 + \frac{1}{2}s^2(s^2 - t^2) \quad (4.17)$$

$$-\frac{1}{5}st^5+\frac{2}{9}st^3+\frac{1}{2}st^2+\frac{3}{10}st$$

For $s,t \in [0,1]$ and with supplementary conditions

$$u_1(s,0)=s, \quad u_2(s,0)=s^2, \quad t\in[0,1] \tag{4.18}$$

Which has the exact solutions

$$u_1(s,t)=s+t, \quad u_2(s,t)=s^2-t^2 \tag{4.19}$$

We applied the method presented in this paper and solved Eq.(4.17). Hence, the Lagrange multiplier is

$$\lambda_1=-1, \quad \lambda_2=\tau-t. \tag{4.20}$$

The iteration formula for this example is

$$u_{1,n+1}(s,t)=u_{1,n}(s,t)-\int_0^t \frac{\partial u_{1,n}(s,\tau)}{\partial t}-\frac{s}{2}\frac{\partial^2 u_{2,n}(s,\tau)}{\partial\tau^2} + \int_0^1 s\tau\left(u_{1,n}^2(x,\tau)+u_{2,n}(x,\tau)\right)dx-1-\frac{2}{3}s\tau-s\tau^2+s]d\tau \tag{4.21}$$

Table 1: numerical results for Example 1

u(s,t)	*e(s,t)*	*e(s,t)*	*e(s,t)*
	n=6	*n=8*	*n=10*
$u_1(s,t)$			
(0.1,0.1)	1.192141×10^{-7}	2.251874×10^{-10}	2.655298×10^{-12}
(0.3,0.3)	5.252154×10^{-7}	3.365699×10^{-9}	1.285213×10^{-12}
(0.6,0.6)	1.258495×10^{-6}	1.215114×10^{-9}	8.521632×10^{-11}
(0.8,0.8)	2.525469×10^{-6}	1.256421×10^{-8}	6.251524×10^{-11}
(1.1)	2.028632×10^{-7}	3.201542×10^{-7}	1.100255×10^{-10}
$u_1(s,t)$			
(0.1,0.1)	1.025145×10^{-8}	3.369872×10^{-11}	6.254123×10^{-12}
(0.3,0.3)	5.231541×10^{-7}	4.201520×10^{-9}	1.021410×10^{-10}
(0.6,0.6)	3.221432×10^{-7}	2.002158×10^{-9}	1.951254×10^{-10}
(0.8,0.8)	1.259820×10^{-6}	2.252189×10^{-8}	1.642598×10^{-9}
(1,1)	1.121541×10^{-7}	1.002314×10^{-8}	2.782184×10^{-9}

$$u_{1,n+1}(s,t)=u_{1,n}(s,t)+\int_0^t(\tau-t)\frac{\partial^2 u_{1,n}(s,\tau)}{\partial t^2}-\frac{s^2}{2}\frac{\partial u_{2,n}(s,t)}{\partial t} + \int_0^1 s(u_{1,n}(x,\tau)-u_2^2(x,\tau))dx+s^2\tau + s\tau^4-\frac{2}{3}s\tau^2-s\tau-\frac{3}{10}s]d\tau \tag{4.22}$$

We begin with an initial arbitrary approximation $u_{1,0}(s,t)=s$, $u_{2,0}(s,t)=s^2$ and using the iteration formula (4.21), (4.22), we get u_n for $n=6,8,10$, and the error function $|u(s,t)-u_n(s,t)|$ and table 2 shows the numerical results obtained by this approximation.

5 Conclusion

In this paper, He's variational iteration method was employed successfully for solving nonlinear systems of the partial mixed Volterra-Fredholm integro-differential equations. This method solves the problem without any need for discretization of the variables, as the numerical results showed, the proposed method is accurate effective method to solve systems of the partial mixed Volterra-Fredholm integro-differential equations.

Table 2: numerical results for Example 2

u(s,t)	*e(s,t)*	*e(s,t)*	*e(s,t)*
	n=6	*n=8*	*n=10*
$u_1(s,t)$			
(0.1,0.1)	1.254744×10^{-12}	1.223589×10^{-14}	5.258847×10^{-15}
(0.3,0.3)	3.20154×10^{-12}	2.33692×10^{-13}	1.02548×10^{-14}
(0.6,0.6)	3.12054×10^{-11}	2.63692×10^{-14}	1.25548×10^{-15}
(0.8,0.8)	4.10941×10^{-12}	4.25899×10^{-10}	2.65938×10^{-13}
(1.1)	8.20541×10^{-10}	6.21314×10^{-8}	2.28984×10^{-12}
$u_1(s,t)$			
(0.1,0.1)	1.12941×10^{-10}	4.20589×10^{-8}	2.65948×10^{-14}
(0.3,0.3)	3.21254×10^{-8}	2.36092×10^{-13}	1.25548×10^{-13}
(0.6,0.6)	3.21354×10^{-8}	2.32592×10^{-12}	1.25548×10^{-14}
(0.8,0.8)	1.19541×10^{-7}	4.25089×10^{-12}	2.65998×10^{-12}
(1,1)	7.27541×10^{-8}	6.20314×10^{-11}	2.10184×10^{-11}

References

1. H. Brunner, *Collocation methods for Volterra Integral and Related Functional Equations*, Cambridge university press, 2004.
2. E. Babolian, K .Maleknejad, M. Roodaki, H. Almasieh, *Two-dimensional triangular functions and their applications to nonlinear 2D Volterra-Fredholm integral equations*, computers and Mathematics with Applications, 60 (2010) 1711-1722.
3. I. Aziz, S.u. Islam, *New algorithms for the numerical solution of nonlinear fredholm and Volterra integral equations using Haar wavelets*, Journal of Computational and Applied Mathematics 239 (2013) 33-345.
4. E. Banifatemi, M. Razzaghi, S. Yousefi, *Two-dimensional Legendre Wavelets Method for the Mixed Volterra-Fredholm Integral Equations*, Journal of Vibration and Control, 13(11) (2007) 1667-1675.
5. J. Biazar, B. Ghanbari, M. Porshokouhi, M. G. Porshokouhi, *He's homotopy perturbation method: A strongly promising method for solving nonlinear systems of the mixed Volterra-Fredholm integral equations*, Computers and Mathematics with Applications, 61 (2011) 1016-1023.
6. K. Maleknejad, K. Mahdiani, *Solving nonlinear mixed Volterra-Fredholm integral equations with the two dimensional block-pulse functions using direct method*, Commun. Nonlinear Sci. Numer. Simulat, 16 (2011) 3512-3519.
7. S.A. Yusefi, A. Lotfi and M. Dehghan, *He's variational iteration method for solving nonlinear mixed mixed Volterra-Fredholm integral equations*, Comput. Math. Appl, 58 (2009 2172-2176.
8. S. Nematia, P.M. Limab, Y. Ordokhani, *Numericla solution of class of two-dimensional nonlinear mixed Volterra integral equations using Legendre polynomials*, Journal of Computational and Applied Mathematics 242 (2013) 5369.
9. E. Babolian, N. Dastani, *He's homotopy perturbation method: An effective tool for solving a nonlinear system of two-dimentional mixed Volterra-Fredholm integral equations*, Mathematical and Computer Modelling, 55 (2012) 1233-1244.
10. J.H. He, *approximate solution of nonlonear differential equations with convolution product nonlinearities*, Comput. Methods Appl. Mech. Eng, 167 12 (1998) 68-73.
11. J.H. He, *Variational iteration method for autonomous ordinary differential systems*, Appl. Math. Comput, 114 (2000) 115-123.
12. J. Biazar, M.G. Porshokouhi, B. Ghanbari, M.G. Porshokouhi, *Numerical solution of functional integral equatoins by the variational iteration mehtod*, Journal of Computational and Applied Mathematics, 235 (2011) 2581-2585.
13. M. Javidi, A. Golbabai, *Exact and numerical solitary wave solutions of generalized Zakharov equation by the variational iteration method*, Chaos Solutions Fractals,36 2 (2008) 309-313.
14. S. Berkani, F. Manseur, A. Maidi, *Optimal control based on the variational iteration method*, Computers and Mathematics with Applications, 64 4 (2012) 41-61.
15. M. Baghani, M. Fattahib, A. Amjadian, *Application of the variational iteration method for nonlinear free vibration of conservative ascillators*, Scientia Iranica, 19 3 (2012) 513-518.
16. B. Kafasha, A. Delavarkhalafia, S.M. Karbassi, *Application of variational iteration method for HamiltonJacobiBellman equations*, Applied Mathematical Modelling, in press (http://dx.doi.org/10.1016/j.apm.2012.08.13).

6/25/2015