On some Applications of Beta Function in some Statistical Distributions

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Abstract: The study established some applications of beta function in probability and cumulative density functions. Just as the gamma function for integers describe factorials, the beta function also defines a binomial coefficient after adjusting indices. The incomplete beta function is a generalization of the beta function that replaces the definite integral of the beta function with an independent integral. In the study, the graph of beta distribution using MATLAB is shown. Further, the study showed how beta and gamma are related theoretically.

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1. Introduction

The Beta function was first studied by Euler and Legendre and was given its name by Jacques Binet; its symbol B is a Greek capital β rather than the similar Latin capital B. Just as the gamma function for integers describes factorials, the beta function can define a binomial coefficient after adjusting indices (Arfken, 1985). The Beta density function is a very versatile way to represent outcomes like proportions or probabilities. It is defined on the continuum between 0 and 1. There are two parameters which work together to determine if the distribution has a mode in the interior of the unit interval and whether it is symmetrical (Johnson, 2013). Moreover, the incomplete beta function is a generalization of the beta function that replaces the definite integral of the beta function with an independent integral.

Notwithstanding, beta function is used in calculating the probability distributive function of relative wind distributions, modelling the experimental frequency distribution of several meteorological data such as relative sunshine and humidity of a place.

The Beta function was the first known Scattering amplitude in String theory first conjectured by Gabriele Veneziano, an Italian theoretical physicist and a founder of string theory. Gabriele Veneziano, a research fellow at CERN, (an European particle accelerator lab) in 1968, observed a strange coincidence of many properties of the strong nuclear force that are perfectly described by the Euler beta-function, an obscure formula devised for purely Mathematical reasons two hundred years earlier by Leonhard Euler. In the flurry of research that followed, Yoichiro Nambu of the University of Chicago, Holger Nielsen of the Niels Bohr Institute and Leonard Susskind of Stanford University revealed that the nuclear interactions of elementary particles modelled as one-dimensional strings instead of zero-dimensional particles were described exactly by the Euler beta function. This was, in effect, the birth of string theory. The Euler Beta function appeared in elementary particle physics as a model for the scattering amplitude in the "dual resonance model.

2. Probability Density Function Of Beta Distribution

The general formula for the probability density function (pdf) of the beta distribution is: $f(x) = \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{B(\alpha,\beta)(b-a)^{p+q-1}} \quad a \le x \le b;\ \alpha, \beta > 0$

Where α *and* β are the shape parameters, *a* and *b* are the lower and upper bounds respectively of the distribution, and $B(\alpha,\beta)$ is the beta function. The beta function has the formula: $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$

The probability density function of the standard beta distribution for $0 \le x \le 1$ and shape parameters α, β > 0 is a power function of the variable x and of its reflection (1-x) and is defined as follows: $\beta(x) = contant.x^{\alpha-1}(1-x)^{\beta-1}$

i.e. Constant = $\frac{1}{B(\alpha,\beta)} = \frac{1}{B(\alpha,\beta)}$

$$\beta(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \quad \ldots\ldots\ldots\ldots (1)$$

where $0 \le x \le 1$ and $\alpha, \beta > 0$.

$$\frac{1}{B(\alpha,\beta)} = \frac{1}{B(\alpha,\beta)}\beta(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx}$$

$$= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha).\Gamma(\beta)/\Gamma(\alpha+\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

Where Γ(z) is the gamma function, the beta function β is a normalized constant to ensure that the total probability integrates to 1. In the above equation x is a realization (an observed value that actually occurred of a random process x).

The definition includes both ends x=0 and x=1 which is consistent with definitions for other continuous distribution supported on a bounded interval which are special cases of the beta distributions. However many/several other authors including W. Feller choose to exclude the ends x=0 and v=1(such that the ends are not part of the density function. And consider instead 0 > x >1.

Several authors including N. L. Johnson and S. Kotz used the symbol p and q (instead of α and β) for the shape parameters of the beta distribution reminiscent of the symbols traditionally used for the parameters of the Bernoulli distributions. Because the beta distribution approach the Bernoulli distribution in the limit when both shape parameters α and β approaches the value of zero.

A random variable x–beta-distributed with parameters α and β will be denoted by x~beta (α, β).

2.1 Graph Of Beta Distribution

In probability theory and statistics, the beta distribution is a family of continuous probability distribution defined on the interval [0, 1] parameterized by two positive shape parameters denoted by α and β or (*p* and *q)*, that appear as exponents of the random variable and control the shape of the distribution. As a useful distribution one can re-scale and shift to create distributions with a wide range of shapes over any finite range. The beta function can take on different shapes depending on the values of the two parameters. The ability of the beta distribution to take this great diversity of shapes (using only two parameters) is partly responsible for finding wide application for modelling actual measurement.

A beta distribution is a probability distribution function similar to the familiar bell-shaped or normal distribution. A normal distribution is symmetrical and all probabilities range from 0 to 1. A beta distribution is a more general variation of a normal distribution that can have a tighter domain and may be asymmetrical. Two constants "a and b" control the symmetry. The software program MATLAB performs arithmetic, calculus, figure plotting and many more. Hundreds of build-in functions make it a useful tool for students and professionals. One of those built-in functions calculates beta probabilities and plots its resulting distribution.

Illustration

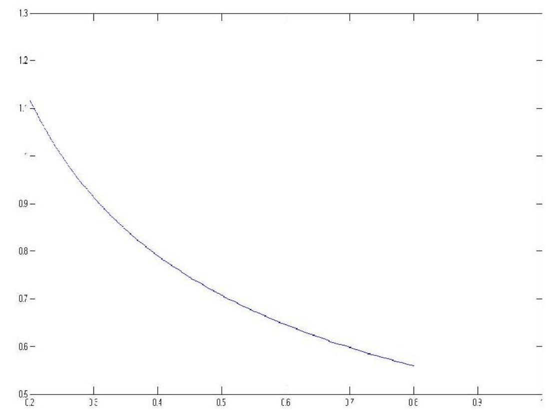
First create an array of 0.2 to 0.8 which contains the domain over the beta distribution spread by inputting the command; x = [0.2:0.1:0.8];

Next, use the values for "a" and "b," the two constants that determine the degree of the distribution symmetry. Thus, inputting the command;

```
const1 = beta (a, b);
a = 0.5;
b = 1;
```

Further, create "y" value for each "x" value to give the probability for each "x" value based on your previous values for "a" and "b" and use the following looping code:

```
for i = 1:length(x);
y (i)=1/const1*x(i)^(a-1)* (1-x(i))^(b-1);
Plot (x, y).
```



3. The Relationship Between Beta And Gamma

The factorial function can be extended to include a non-integer argument through the use of Euler second integral given as: $z! = \int_0^\infty e^{-t} t^z \, dt$

This equation is mostly known as the generalised factorial function.

The classical case of the integer forms the factorial function. i.e. n! consists of the product of n and of all integers less than n down to 1 as follows:

$$n! = \begin{cases} n(n-1)(n-2)\ldots3.2.1 & n = 1,2,3 \\ 1 & n = 0 \end{cases}$$

where by definition 0! = 1

Through a simple translation of the z- variable we can obtain the familiar gamma function as follows:

$$(z) = \int_0^\infty e^{-t} t^{z-1} dt = (z-1)!$$

$$\Rightarrow \Gamma(z) = (z-1)!$$

The relationship between gamma function and the beta function can be demonstrated easily by:

$$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Proof: the relation of the beta function to gamma functions is obtained using integration by parts as follows:

Let U = $y^{\alpha-1}$ and

dv = $(1-y)^{\beta-1}$

$$\frac{du}{dy} = (\alpha - 1)y^{\alpha-2}dy$$

$v = \frac{-1(1-y)^{\beta}}{\beta}$ *implies*,

$$B(\alpha,\beta) = \begin{cases}1\\0\end{cases} y^{\alpha-1}\left(\frac{-(1-y)^{\beta}}{\beta}\right) + \frac{\alpha-1}{\beta}\int_0^1 y^{\alpha-2}(1-y)^{\beta+1}dy$$

$$= 1^{\alpha-1}\left(\frac{-(1-1)^{\beta}}{\beta}\right) - \left[0^{\alpha-1}\left(\frac{-(1-0)^{\beta}}{\beta}\right)\right] + \frac{\alpha-1}{\beta}\int_0^1 y^{\alpha-2}(1-y)^{\beta+1}dy$$

$$= 0 - 0 + \frac{\alpha-1}{\beta}\int_0^1 y^{\alpha-2}(1-y)^{\beta+1}dy = \frac{\alpha-1}{\beta}\beta(\alpha-1,\beta+1)$$

$$= \frac{(\alpha-1)(\alpha-2)(\alpha-3)\ldots}{\beta(\beta+1)\ldots(\beta+\alpha-2)}\beta(1,\beta+\alpha-1)$$

$$= \frac{(\alpha-1)(\alpha-2)\ldots 1}{\beta(\beta+1)\ldots(\beta+\alpha-2)}\int_0^1 (1-y)^{\alpha+\beta-2}dy$$

$$= \frac{(\alpha-1)(\alpha-2)\ldots 1}{\beta(\beta+1)\ldots(\beta+\alpha-2)}\begin{cases}1\\0\end{cases}\frac{-(1-y)^{\alpha+\beta-1}}{\alpha+\beta-1}$$

$$= \frac{(\alpha-1)(\alpha-2)\ldots 1}{\beta(\beta+1)\ldots(\beta+\alpha-2)(\beta+\alpha-1)} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Hence,

$$B(\alpha,\beta) = \int_0^1 y^{\infty-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The Beta function B in the denominator plays the role of a "normalizing constant" which assures that the total area under the density curve equals 1. The Beta function is equal to a ratio of gamma functions. Alternatively, since beta function can be expressed as a ratio of gamma functions;

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

we can substitute into equation above to obtain:

$$\beta(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

Keeping in mind that for integers, $\Gamma(n) = (n-1)!$

One can do some checking and get an idea of what the shape might be.

4. Moments Of The Beta Function

If x has a beta distribution with parameters p and q; it can be shown that its mean μ and variance σ^2 are given by:

$$E(Y) = \mu = \frac{\alpha}{\alpha+\beta} \text{ and } V(Y) = \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Proof

$$E(Y) = \mu = \frac{\alpha}{\alpha+\beta}$$

$$\Rightarrow E(Y) = \int_0^1 y\left(\frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}\right)dy$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 y^1.y^{\alpha-1}(1-y)^{\beta-1}dy$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 y^{\alpha}(1-y)^{\beta-1}dy$$

$$= \frac{1}{B(\alpha,\beta)}B(\alpha+1,\beta) = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

$$B(\alpha+1,\beta) = \frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}$$

and

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\therefore \frac{\Gamma(\alpha+1)\,\Gamma(\beta)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+1+\beta)\,\Gamma(\alpha)\,\Gamma(\beta)} = \frac{\Gamma(\alpha+1)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+1+\beta)\,\Gamma(\alpha)}$$

$$= \frac{(\alpha)!\,(\alpha+\beta-1)!}{(\alpha+\beta)!\,(\alpha-1)!} = \frac{(\alpha)(\alpha-1)!\,(\alpha+\beta-1)!}{(\alpha+\beta)(\alpha+\beta-1)!\,(\alpha-1)!}$$

$$\therefore E(Y) = \frac{\alpha}{\alpha+\beta}$$

Also;

$$V(Y) = \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$V(Y) = E(Y^2) - (E(Y))^2$$

Since;

$$E(Y^2) = \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\frac{(\alpha+1)!}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

Now LCM gives;

$$= \frac{(\alpha+\beta)(\alpha+1)\alpha - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{(\alpha+\beta)(\alpha^2+\alpha) - (\alpha^3+\alpha^2\beta+\alpha^2)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\rightarrow \frac{\alpha^3+\alpha^2+\alpha^2\beta+\alpha\beta-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Illustration

Suppose that the proportion of downtime of UDS machines Y has Beta distribution with α =1 and β = 2. If the cost due to downtime is C = 10+20Y +4Y^2, What is E[C] and V(C)?

Solution

We have Y~beta (1, 2)

$$\therefore E(y) = \frac{\alpha}{\alpha+\beta} = \frac{1}{3}\, also$$

$$E(Y^2) = \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} = \frac{1}{6}$$

$$now\ E(C) = 10 + 20E(y) + 4E(y^2)$$

$$= 10 + \frac{20}{3} + \frac{4}{6} = \frac{52}{3}$$

$$Also\ V(c) = E(C^2) - (E[C]^2)$$

$$but\ (E[C]^2) = E[(10 + 20Y + 4Y^2)]^2$$

$$= E[100 + 400Y + 40Y^2 + 400Y^2 + 80Y^3 + 16Y^4]$$

$$= 100 + 400E[Y] + 440E[Y^2] + 80E[Y^3] + 16E[Y^4]$$

$$100 + \frac{400}{3} + \frac{440}{6} + \frac{80}{10} + \frac{16}{15} = \frac{4736}{15}$$

$$\therefore V(C) = \frac{4736}{15} - \left(\frac{52}{3}\right)^2$$

$$\therefore V(C) = 15.289$$

5. Cumulative Distribution Function The incomplete beta function β_x (α, β) is defined as follows:

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt \qquad where\ 0 \le x \le 1\ ,\alpha > 0, \beta > 0$$

For $x = 1,\ B_x(\alpha, \beta)$ is known as the complete beta function.

The formula for the cumulative distributive function (cdf) of the beta distribution is also called the incomplete beta function ratio (commonly denoted by I_x) and is defined as:

$$F(x) = I_x(\alpha,\beta) = \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)} = \frac{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt}{\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt}, 0 \le t \le 1; \alpha, \beta > 0$$

where ***B*** is the beta function defined above.

6. Applications

6.1 Beta Function And Beta Distribution

A beta distribution is a type of probability distribution that is commonly used to describe uncertainties about the true value of a proportion. There are appropriate distributions to express uncertainties about the prior values for prevalence sensitivity. The beta distribution can be defined by the two parameters, alpha and beta (written as Beta (alpha, beta) with alpha = s+1 and beta = n-s+1. Where s is the number of successes out of n trials. If there is no information on which to base a prior distribution, alpha = beta = 1 should be used. This result is a uniform distribution in which all values between 0 and 1 has equal probability of occurrence. The alpha and beta parameters are best understood in terms of success and failures of an event where α =success +1 and β = failures +1. Therefore zero success and zero failure may be represented as $B(\alpha = 1,\ \beta = 1)$ distributions. Therefore the mean (expectation) of the beta distribution is $\mu = \frac{1}{1+1} = \frac{1}{2} = 0.5$. Based on zero success and failure the expectation of a success has a 50% probability (p=0.5).

The Beta can be used to describe not only the variety observed across people, but it can also describe your subjective degree of belief (in a Bayesian sense). For example, how likely is it that Ghana would be independent in 2016? My brother thinks the probability is 0.11, my sister thinks the chances are 0.33, and I think the chances are 0.12. If you are not entirely sure that the probability is say 0.51, but rather you think, that is the most likely value, and that there is some chance that the value is higher or lower, then your personal beliefs can be described as a Beta distribution.

Also imagine a new born baby who observed the sun on his first day. After the sunset the new born has uncertainty as to whether or not he will see the sun again. Those uncertainties are represented with a beta distribution. The Beta distribution can be used to model events which are constrained to take place within an interval defined by a minimum and maximum value. For this reason, the Beta distribution along with the triangular distribution is used extensively in PERT, critical path method (CPM) and other project management or control systems to describe the time of completion of a task. In project management, shorthand computations are widely used to estimate the mean and standard deviation of the Beta distribution.

6.2 Beta Function And Beta Probability Distribution

The beta distribution is a conjugate prior (meaning it has the same functional form therefore also often called convenient prior) to the binomial likelihood function in the Bayesian inferences and as such is often used to describe the uncertainty about a binomial probability. Given that a number of trials n have been made with the number of recorded successes, alpha is a set of value (s+x) and beta is (n-s+y) where s is the number of successes out of n trials. And where Beta (x, y) is the prior.

Illustration

Bank may have data on the number of creditors that have defaulted (s) out of the total number of creditors (n) of that type. Then the probability that the next creditor of the same type will default can be estimated as Beta (s+1, n-s+1) where n is the number of trials and s the number of successes.

Thus, a random survey of 100 vehicle owners over 65years of age reveals that 57 considered a newly proposed insurance policy to be more attractive than their current policy. We estimate that the fraction of drivers in this age group who would have the same opinion is; Beta (s+1, n-s+1) where S = 57 and n =100.

Then Beta (57+1, 100-57+1) = Beta (58, 44)

6.3 Beta Function In Risk Analysis

In risk analysis, if we are sure that data is collected according to a binomial process, we can use the beta distribution to describe our uncertainty about the proportion or probability by applying the formula: $p = Riskbeta\,(s+1, n-s+1)$ where n is the number of trials of sample and s is the number of successes.

Population prevalence of (or fraction of animal with) a disease. Imagine that we have some data on the prevalence of BSE (Bovine Spongiform Encephalopathy) among cows in some country. Out of

3000 calves, 300 were randomly selected, and tested examining brain tissue. 6 were found to be infected. Assuming for the moment that the test is 100% accurate (i.e. all those with BSE found in the brain tissue had BSE and none of the others did), we could get a single point (best guessed) estimate of the prevalence of BSE among the calve population in general as $\frac{6}{300} = \frac{1}{50}\ or\ 2\%$. However, the relatively small number of sample taken means that there remains some reasonable uncertainty about what that true prevalence actually is. Taken a small (relative to the population size) random sample from a population and then determining whether each sample either has some particular characteristic is an example of a binomial process. Thus, the prevalence is equivalent to the binomial probability and we can therefore use the beta distribution to describe the remaining uncertainty about p, as follows:

$$p = Riskbeta(6+1, 300-6+1)$$
$$= Riskbeta(7, 295)$$

If we had tested 1000 animals and had found the same proportion (2%) infected, our estimate of the population prevalence would have been more precise (narrower distribution of uncertainty).

Conclusion

In conclusion, The Study demonstrated some possible applications of beta function giving its historical perspective. Further, we used beta function in solving some statistical distributions such as probability density function and the cumulative distribution function. The study also established a relationship between beta and gamma function.

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