A MODIFIED EXPONENTIAL WEIBUL LOG-LOGISTIC POISSON DISTRBUTION WITH ITS PROPERTIES AND ANALYTIC APPROACH

Adeyemi-Gidado S.A., Akomolafe A.A. and Ojo, O. O

Department of Statistics, Federal University of Technology Akure, Nigeria.
ajibolasarat@gmail.com, aaakomolfe@futa.edu.ng, ooojo@futa.edu.ng

Abstract: In an effort to address for the family of distribution that permits flexibility in simulating realworld phenomena, a new family of univariate probability distribution called exponential Weibull log-logistic Poisson family of probability distribution is introduced in this paper by compounding. the T- R [Y] family of distribution. In the field of reliability analysis, the selection of an appropriate lifespan model is critical. With a multitude of lifetime distributions accessible, the hunt for a more suited distribution remains essential. In this paper, a new model of a life time Distribution was developed that mainly generalizes these distributions.Parameter estimation of the four parameters of this distribution are studied. Simulation study was carried out to establish the validity of the model , thus showing that the modified exponentiated Weibull log-logistic Poisson distribution can be used quite effectively infitting and analyzing real lifetime data. A unique class of distributions derived from the notion of exponential generalization, improved with changes to boost flexibility. Our suggested distribution incorporates multiple hazard rate profiles, giving enhanced flexibility. The new distribution has the advantage of being capable of modeling various shapes of ageing and failure criteria. We derive several of its structural properties including moment, survival function, hazard function and order statistics. The new density function can be expressed as a linear mixture of exponentiated Weibull densities. We proposed a linear regression model using a new distribution–the exponential Weibull log-logistic Poisson distribution. The maximum-likelihood method is used to estimate the model parameters and simulation results were provided to assess the performance of the proposed maximum likelihood procedure. The results of this study offer a strong basis for real-life applications.
[Adeyemi-Gidado S.A., Akomolafe A.A. and Ojo, O. O. A MODIFIED EXPONENTIAL WEIBUL LOG-LOGISTIC POISSON DISTRBUTION WITH ITS PROPERTIES AND ANALYTIC APPROACH. *Academ Arena* 2026;18(2):131-142]. ISSN 1553-992X (print); ISSN 2158-771X (online). http://www.sciencepub.net/academia. 03. doi:10.7537/marsaaj180226.03

Key words: Univariate Probability Distribution; survival function; Weibull distribution; maximum-likelihood; exponentiated Weibull densities.

1. INTRODUCTION

There are several typical theoretical distributions documented in the literature, including the Geometric distribution, Exponential distribution, Uniform distribution, Normal distribution, Gamma distribution, Beta distribution, and others. These standard distributions are well recognised for their significant relevance and extensive use in several fields of study. Various lifespan distributions, including exponential, Weibull, Gompertz, and Gumbel distributions, have been employed for simulating dependability, human mortality, and actuarial data. The Exponential distribution is commonly used to analyse lifespan data in numerous fields of research because to its simplicity and analytical tractability (Maurya, Kaushik, Singh & Singh, 2016). In statistical modeling and data analysis, probability distributions are essential tools for representing and interpreting real-world occurrences. Probability distribution functions generally are used in describing the real-world phenomenal and are classified in to discrete and continuous distribution where the discrete and continuous distribution, where the discrete is called probability mass function (pmf) and the continuous distribution is known as probability density
function (pdf). Various methods of generating continuous and discrete univariate distributions have been put forward. These methods include the method based on differential equations (Pearson (1895) and Bur (1942)), method based on transformation (Johnson (1949), method based on quantiles (Turkey (1960), Aldeni et al, (2017), (Azzadini (1895)), method of addition of parameter(s) and generalization (Mudholkar and Sristpawa (1992), Marshal and Oikin (1997), Shaw and Buckey (2009)), and for the discrete univariate distribution (Adamidis and Loukas (1998)), method based on generators (Eugene et al (2002), Jones (2009), Cordeiro and de Castro (2011)), Method based on the composition of densities (Corray and Anada (2005)) and the Tramsformed-Transformer method (Alzatreh et al (2013), Alzatreh et al (2014)).
The Weibull distribution is a very popular model in statistical modeling and has been extensively used over the past decades for modeling data in engineering,

reliability and biological studies. The need for extended forms of the Weibull model arises in many applied areas. Some of the classical extensions of the Weibull model with applications were studied (Murthy et al (2004); Nadrajah and Kortz (2006)). There has been an increased interest among statisticians to develop new methods for generating new families of distributions, because there is a persistent need for extending the classical forms of the well- known distributions to be more capable for modeling data in different areas and fields such as life-time analysis in engineering, economics, finance, demography, actuarial, biological and medical sciences.

The aim of this research is to develop a new lifetime model the exponential Weibull loglogistic Poisson distribution (EWLLP) with a proposed regression model, using the T-R[Y] family of distributions, we construct the five-parameter EWLLP model and give a comprehensive description of some of its mathematical properties.

The proposed distribution contains several lifetime distributions, such as EW (Mudheikar et al,1995; 1996), Weibull (1951), OLL-Weibull (Cocray, 2006), exponentiated exponential (Guptaand Kundu, (2001)), exponential and Rayleigh distributions among others as a special case. We are motivated to introduce the EWLLP distribution because

(i) it contains a number of aforementioned of known lifetime sub-models;

(ii) the EWLLP distribution exhibits monotone as well as non-monotone hazard rates which makes this distribution to be superior to another lifetime distributions which exhibit only monotonically increasing/decreasing or constant hazard rates

(iii) the EWLLP distribution can be viewed as a mixture of exponentiated Weibull distribution introduced by Mudheikar et al, (1995).

(iv) It can also be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other common distributions and

(v) the EWLLP distribution outperforms several competitive distributions.

2.0 Development of the Exponential Weibull Log-Logistic Poisson Distribution (EWLLP)

The probability density function (pdf) and cumulative distribution function (cdf) of the EW distribution are given by the following:

Suppose T follows exponential distribution with cdf, pdf and quantile function given as;

$$F_T(x) = 1 - e^{-\lambda x}$$
$$f_T(x) = \lambda e^{-\lambda x}$$
$$Q_T(x) = \frac{-log(1-p)}{\lambda}$$

Let Y be a random variable of log-logistics distribution with cdf, pdf and quantile function express as;

$$F_Y(x) = 1 - (1+x)^{-1}$$
$$f_y(x) = (1+x)^{-2}$$
$$Q_y(p) = \frac{p}{1-p}$$

Suppose T, R and Y are random variables with respective cumulative distribution function (cdf) $F_T(x) = P(T \le x)$, $F_{R\,(x)} = P(R \le x)$ and $F_Y(x) = P(Y \le x)$

Suppose the corresponding densities of T, R and Y exist and denote them by $f_{T\,(x)}$, $f_{R\,(x)}$ and $f_y(x)$.

Assume that $T \in (a, b)$ *and* $Y \in (c, d)$ for $-\infty \le a < b \le \infty$ *and* $-\infty \le c < d \le \infty$, then the T-R(Y) family of distribution was defined by the cdf:

$$F_x(x) = \int_a^{Q_y(F_R(x))} f_T(t)dt = F_T\left(Q_y\left(F_R(x)\right)\right), \ x \in R \quad (1)$$

$$F_x(x) = F_T\left(\frac{F_R(x)}{1-F_{R(x)}}\right) = 1 - e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

$$F_x(x) = 1 - e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)} e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

The corresponding probability density function (pdf) of the cdf in (1) was given by:

$$f_x(x) = f_R(x) \times \frac{f_T\left(Q_y\left(F_R(x)\right)\right)}{f_y\left(Q_y\left(F_R(x)\right)\right)}, \ x \in R \quad (2)$$

$$= f_R(x) \times \frac{\lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}}{1 + \left(\frac{F_R(x)}{1+F_{R(x)}}\right)^{-2}}$$

2.1 Expansions for the Densities of Exponential Weibull Log-Logistic Poisson (EWLLP) Distribution

T-R(Y) – P family is one of the sub-families of the T-R(Y)-PS family of distribution and the cumulative distribution function is stated below

$$F_{T-R(y)-p} = 1 - \frac{e^{\theta}(1 - F_T(|Q_y(F_R(x))))_{-1}}{e^{\theta}{-1}} \quad , x \in R \qquad (3)$$

But

$$F_T\left(Q_y\left(F_R(x)\right)\right) = 1 - e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}$$

$$1 - F_T\left(Q_y\left(F_R(x)\right)\right) = 1 - 1 + e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}$$

$$1 - F_T\left(Q_y\left(F_R(x)\right)\right) = e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}$$

$$F_{T-R(Y)-p} = 1 - \frac{e^{\theta}(1 - F_T(|Q_y(F_R(x))))_{-1}}{e^{\theta}{-1}}$$

$$F_{T-R(y)-p} = 1 - \frac{e^{\theta e^{-\lambda\left[\frac{F_R(x)}{1-F_R(x)}\right]}} - 1}{e^{\theta} - 1}$$

The corresponding probability density function (pdf) is given by:

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}{-1}} \text{ x } \frac{f_T\left(Q_y(F_R(x))\right)}{f_y\left(Q_y(F_R(x))\right)} e^{\theta}\left(1 - F_T(Q_y\left(F_R(x)\right))\right) \qquad (4)$$

$$f_T\left(Q_y\left(F_{(x)}\right)\right) = \lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}$$

$$f_y\left(Q_y\left(F_R(x)\right)\right) = 1 + \left(\frac{F_R(x)}{1 - F_R(x)}\right)^{-2}$$

$$1 + \left(\frac{F_R(x)}{1 - F_R(x)}\right)^{-2} = \left(\frac{1 - F_R(X) + F_R(x)}{1 - F_R(x)}\right)^{-2}$$

$$= \left(\frac{1}{1 - F_R(x)}\right)^{-2} = \left(1 - F_R(x)\right)^2$$

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}{-1}} \text{ x } \frac{f_T\left(Q_y(F_R(x))\right)}{f_y\left(Q_y(F_R(x))\right)} e^{\theta}\left(1 - F_T(Q_y\left(F_R(x)\right))\right)$$

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}{-1}} \text{ x } \frac{\lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}}{1 + \left(\frac{F_R(x)}{1+F_R(x)}\right)^{-2}} e^{\theta e^{-\lambda\left[\frac{F_R(x)}{1-F_R(x)}\right]}}$$

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}{-1}} \text{ x } \frac{\lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}}{\left(1 - F_R(x)\right)^2} e^{\theta e^{-\lambda\left[\frac{F_R(x)}{1-F_R(x)}\right]}} \qquad (5)$$

$$F_{T-R(y)p}(x) = 1 - \frac{exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right] - 1}{(e^{\theta} - 1)} \theta, x > 0 \qquad (6)$$

$$f_{T-R(y)p}(x) = \frac{\theta f_R(x)}{e^{\theta} - 1} \times \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right]}{[1 - F_R(x)]^2} \qquad (7)$$

Let R be a Weibull Distribution with the cumulative distribution function (CDF) given below as:

$$F(x) = 1 - e^{-(x/\alpha)^{\beta}} \quad , \quad x \geq 0$$

with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$ .The probability density function (PDF) is given as:

$$f(x, \alpha, \beta) = \frac{\beta}{\alpha}(x/\alpha)^{\beta-1} e^{-(x/\alpha)^{\beta}}$$

But,

$$F(x) = F_R(x) = 1 - e^{-(x/\alpha)^{\beta}} \quad , \quad x \geq 0$$

Now, to find $1 - F_R(x)$

$$1 - F_R(x) = 1 - (1 - e^{-(x/\alpha)^{\beta}}) \quad (8)$$

$1 - F_R(x) = e^{-(x/\alpha)^{\beta}}$

Compute the Ratio,

$$\frac{F_R(x)}{1 - F_R(x)} = \frac{1 - e^{-(x/\alpha)^{\beta}}}{e^{-(x/\alpha)^{\beta}}} \quad (9)$$

Simplifying the expression inequation (9),

$$\frac{F_R(x)}{1 - F_R(x)} = e^{(x/\alpha)^{\beta}} - 1 \quad (10)$$

Substitute the expression into equation (6)

$$F_{T-R(y)p}(x) = 1 - \frac{exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right] - 1}{(e^{\theta} - 1)} \theta, x > 0 \quad (11)$$

We have:

$$F_{EWLLP}(x) = 1 - \frac{exp\left[\theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right)}\right] - 1}{(e^{\theta} - 1)} \quad (12)$$

2.2 Simplification of the PDF

$f_R(x) = \frac{\beta}{\alpha}(x/\alpha)^{\beta-1} e^{-(x/\alpha)^{\beta}}$, x≥0

$$f_{T-R(y)p}(x) = \frac{\theta f_R(x)}{e^{\theta} - 1} \times \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right]}{[1 - F_R(x)]^2} \quad (13)$$

where $\theta, \lambda, x > 0$

$F_R(x) = 1 - e^{-(x/\alpha)^{\beta}}$, x≥0

$1 - F_R(x) = e^{-(x/\alpha)^{\beta}}$

$\frac{F_R(x)}{1 - F_R(x)} = e^{(x/\alpha)^{\beta}} - 1$

$$e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} = e^{-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right)}$$

$[1 - F_R(x)]^2 = (e^{-(x/\alpha)^{\beta}})^2 = (e^{-2(x/\alpha)^{\beta}}$

Substitute into equation (3.7)

$$f_{T-R(y)p}(x) = \frac{\theta f_R(x)}{e^{\theta} - 1} \times \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right]}{[1 - F_R(x)]^2}$$

$$= \frac{\theta \frac{\beta}{\alpha}(x/\alpha)^{\beta-1} e^{-(x/\alpha)^{\beta}}}{e^{\theta} - 1} \times \frac{\lambda\theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right)} e^{\theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right)}}}{(e^{-2(x/\alpha)^{\beta}})}$$

$$f_{EWLLP}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta} - 1)}(x/\alpha)^{\beta-1} e^{(x/\alpha)^{\beta}} \times e^{\left[-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right) + \theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right)}\right]} \quad (15)$$

2.3 Properties and Extensions of Exponential Weibull Log-Logistic Poisson Distri

2.3.1 Moment of Exponential Weibull Log-Logistic Poisson (EWLLP) Distribution

Recall the pdf from equation (15) where we have

$$f_{EWLLP}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta} - 1)}(x/\alpha)^{\beta-1} e^{(x/\alpha)^{\beta}} \times e^{\left[-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right) + \theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}} - 1\right)}\right]}$$

Let U = $(x/\alpha)^{\beta}$,

Using the binomial expansion and Taylor's Series, we then have

$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$ and $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, -\infty < x < \infty$

$$f_{T-R(y)p}(x) = \frac{\theta\,\beta\lambda}{\alpha} (x/_\alpha)^{\beta-1} \sum_{t=0}^{\infty}\sum_{i}^{\infty}\sum_{j}^{\infty}\sum_{k}^{\infty}\sum_{r}^{\infty} \left(\frac{\lambda}{i!}\right)^i (-1)^{n-j} \binom{n}{j} \frac{\theta^k}{n!} \frac{(k\lambda)^n}{n!}\, e^{-[t\theta-i-j-n](x/_\alpha)^\beta}$$

Let $(Z_i) = \sum_t^\infty \sum_i^\infty \sum_j^\infty \sum_k^\infty \sum_r^\infty \left(\frac{\lambda}{i!}\right)^i (-1)^{n-j} \binom{n}{j} \frac{\theta^k}{n!} \frac{(k\lambda)^n}{n!}$

$f_{T-R(y)p}(x) = \frac{\theta\,\beta\lambda}{\alpha^\beta} (x)^{\beta-1} Z_i e^{-[t\theta-i-j-n](x/_\alpha)^\beta}$ To obtain the r^th moment of the distribution

$$\mu^r = E[x^r] = \int_0^\infty x^r f(x)dx$$

Where f(x) is pdf of EWLLP distribution in equation

$E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \int_{-\infty}^{\infty} x^r f_{T-R(y)-p}(x)dx$

$= \int_{-\infty}^{\infty} x^r \frac{\theta\,\beta\lambda}{\alpha^\beta} x^{\beta-1} Z_i e^{-[t\theta-i-j-n](x/_\alpha)^\beta} dx$

$= \frac{\theta\,\beta\lambda}{\alpha^\beta} Z_i \int_{-\infty}^{\infty} x^r.\ x^{\beta-1} e^{-[t\theta-i-j-n](x/_\alpha)^\beta} dx$

$= \frac{\theta\,\beta\lambda\, Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} x^{r+\beta-1}.\ e^{-[t\theta-i-j-n](x/_\alpha)^\beta} dx$

Let $U = [t\theta-i-j-n](x/_\alpha)^\beta$

$E(x^r) = \frac{\theta\,\beta\lambda\, Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} x^{r+\beta-1}.\ e^{-U} dx$

$U = [t\theta-i-j-n](x/_\alpha)^\beta$

$\left|\frac{U}{[t\theta-i-j-n]}\right|^{\frac{1}{\beta}} = (x/_\alpha)$

$X = \frac{\alpha U^{\frac{1}{\beta}}}{\beta(t\theta-i-j-n)^{1/_\beta}}$

$\frac{dx}{du} = \frac{\alpha U^{1/_\beta-1}}{\beta(t\theta-i-j-n)^{1/_\beta}}$

$dx = \frac{\alpha U^{1/_\beta-1}}{(t\theta-i-j-n)^{1/_\beta}} du$

$E(x^r) = \frac{\theta\,\beta\lambda\, Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} x^{r+\beta-1}.\ e^{-U} dx$

$= \frac{\theta\,\beta\lambda\, Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} \left(\frac{\alpha U^{\frac{1}{\beta}}}{(t\theta-i-j-n)^{1/_\beta}}\right)^{r+\beta-1} .\ e^{-U} . \frac{\alpha U^{1/_\beta-1}}{\beta([t\theta-i-j-n])^{1/_\beta}} du$

$= \frac{\theta\,\beta\lambda\, Z_i}{\alpha^\beta} . \frac{\alpha^{r+\beta-1}}{\left[(t\theta-i-j-n)^{1/_\beta}\right]^{r+\beta-1}} . \frac{\alpha}{\beta([t\theta-i-j-n])^{1/_\beta}} \int_{-\infty}^{\infty} U^{\frac{1}{\beta}(r+\beta-1)} . U^{1/_\beta-1} . e^{-U} du$

But, $U^{\frac{r}{\beta}} e^{-U} du = \Gamma r/\beta + 1$

$E(x^r) = \frac{\theta\,\lambda\alpha^r Z_i \Gamma r/\beta+1}{(t\theta-i-j-n)^{\frac{r}{\beta}+1}}$ (16)

From equation (16), the following four moments were derived

$\mu = E(x^1) = \frac{\theta\,\lambda\alpha^1 Z_i \Gamma 1/\beta+1}{(t\theta-i-j-n)^{\frac{1}{\beta}+1}}$

$\mu_2' = E(x^2) = \frac{\theta\,\lambda\alpha^2 Z_i \Gamma 2/\beta+1}{(t\theta-i-j-n)^{\frac{2}{\beta}+1}}$

$$\mu_3' = E(x^3) = \frac{\theta\,\lambda\alpha^3 Z_i \Gamma\, 3/\beta + 1}{(t\theta-i-j-n)^{\frac{3}{\beta}+1}}$$

$\mu_4' = \mathrm{E}(x^4) = \frac{\theta\, \lambda\alpha^4 Z_i \Gamma 4/\beta+1}{(t\theta-i-j-n)^{\frac{4}{\beta}+1}}$

Variance = E $(x^2) - \left(\mathrm{E}\,(x)\right)^2$

$$= \frac{\theta\, \lambda\alpha^2\, Z_i\Gamma 2/\beta+1}{(t\theta-i-j-n)^{\frac{2}{\beta}+1}} - \left(\frac{\theta\, \lambda\alpha^1 Z_i\Gamma 1/\beta+1}{(t\theta-i-j-n)^{\frac{1}{\beta}+1}}\right)^2 \qquad (17)$$

2.3 .2 The Survival Function

The derivation is as shown below

$$S(x) = 1 - F(x)$$

$$F_{EW(LL)P}(x) = 1 - \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}-1}{e^{\theta}-1} \qquad (18)$$

$$f_{E-W(LL)P}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta}-1)}(x/\alpha)^{\beta-1} e^{(x/\alpha)^\beta} \times e^{\left[-\lambda\left(e^{(x/\alpha)^\beta}-1\right)+\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}$$

$$S(x) = 1 - F(x)$$

$$= 1 - \left\{1 - \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}-1}{e^{\theta}-1}\right\}$$

$$= 1 - 1 + \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}-1}{e^{\theta}-1}$$

$$S(x) = \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}-1}{e^{\theta}-1} \qquad (19)$$

2.3.3 The Hazard Function

$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} =$

$$= \frac{\frac{\theta\beta\lambda}{\alpha\left(e^{\theta}-1\right)}(x/\alpha)^{\beta-1} e^{(x/\alpha)^\beta} \times e^{\left[-\lambda\left(e^{(x/\alpha)^\beta}-1\right)+\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}}{\frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}-1}{e^{\theta}-1}} \qquad (20)$$

$$h(x) = \frac{\theta\beta\lambda}{\alpha}(x/\alpha)^{\beta-1} e^{(x/\alpha)^\beta} \times e^{\left[-\lambda\left(e^{(x/\alpha)^\beta}-1\right)\right]} \qquad (21)$$

2.3.4 Order Statistics

Let $X_1, X_2 \ldots\ldots\ldots\ldots X_n$ be a random sample of size n from the distribution and Let $X_{(1)}, X_{(2)} \ldots\ldots\ldots\ldots X_{(n)}$ denote the corresponding order.

Let F(x) be the cumulative distribution function (CDF) corresponding to f(x)

Then, the density function of the order statistics $X_{(k)}$ is given as:

$$f_{(k)}(x) = \frac{n!}{(k-1)!\,(n-k)!} \cdot (F_X(x))^{K-1} \cdot (1 - F_X(x))^{n-k} \cdot f_x(x)$$

$$= \frac{\theta}{e^{\theta}-1} \cdot \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} e^{\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}}, f_{R(x)}}{(1-F_{R(x)})^2}$$

$$f_{T-R(y)-p}(x) \quad = \int_0^x f(t)dt$$

$$f_{(k)}(x) = \frac{n!}{(k-1)!\,(n-k)!} \cdot (F_X(x))^{K-1} \cdot (1-F_X(x))^{n-k} \cdot f_x(x)$$

$$= \frac{n!}{(k-1)!(n-k)!} \cdot \left(\int_0^x f(t)_x dt\right)^{K-1} \cdot \left(1-\int_0^x f(t)_x dt\right)^{n-k} \cdot \frac{\theta}{e^{\theta}-1} \cdot \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} e^{\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}}, f_{R(x)}}{(1-F_{R(x)})^2}$$

$$= \frac{n!}{(k-1)!(n-k)!} \cdot \left(\int_0^x \frac{\theta}{e^{\theta}-1} \cdot \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)} e^{\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}}, F_{R(x)}}{(1-F_{R(t)})^2} f_R(t)dt\right)^{K-1} \cdot$$

$$\left(1-\int_0^x \frac{\theta}{e^{\theta}-1} \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)} e^{\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}}, F_{R(x)}}{(1-F_{R(t)})^2} f_R(t)dt\right)^{n-k} \cdot \frac{\theta}{e^{\theta}-1} \cdot \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(tx)}\right)} e^{\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}}, f_{R(x)}}{(1-F_{R(t)})^2} \quad \textbf{(23)}$$

3.0 Method of Estimating the parameters using MaximumLikelihood Estimation technique

Suppose $X_1 ... X_n$ be a random sample from the EWLLP distribution with an unknown parameter (α, β, θ, λ and θ). The pdf of EGEL is given as:

$$f_{E-W(LL)P}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta}-1)} (x/\alpha)^{\beta-1} e^{(x/\alpha)^{\beta}} \times \; e^{\left[-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)+\theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)}\right]}$$

The likelihood function is given as:

$$\mathcal{L}(\alpha,\beta,\theta,\lambda) = \prod_{i=1}^{n} f(x_i,\alpha,\beta,\theta,\lambda)$$

Therefore, thelikelihood function is:

$$\mathcal{L}(\alpha,\beta,\theta,\lambda) = \sum_{i=0}^{n} f(x_i,\alpha,\beta,\theta,\lambda)$$

$$\ell = \sum_{i=0}^{n} \left[\ln\theta + \ln\beta + \ln\lambda - \ln\alpha - \ln\left(e^{\theta}-1\right) + (\beta-1)\ln\frac{x_i}{\alpha} - \left(\frac{x_i}{\alpha}\right)^{\beta} - \lambda\left(e^{\left(\frac{x_i}{\alpha}\right)^{\beta}}-1\right) + \theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)}\right]$$

Solve for $\alpha, \beta, \theta, \lambda$ and insert them to the above equation

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=0}^{n}\left[-\frac{1}{\alpha} - \frac{(\beta-1)}{\alpha} - \frac{\beta U_i}{\alpha} + \frac{\lambda\beta U_i e^{U_i}}{\alpha} - \frac{\theta\lambda\beta U_i e^{U_i S_i}}{\alpha}\right]$$

Derivation with respect to β

$$\frac{\partial\ell}{\partial\beta} = \sum_{i=0}^{n}\left[\frac{1}{\beta} + \ln\frac{x_i}{\alpha} - \frac{\partial U_i}{\partial\beta} - \frac{\lambda\partial(e^{U_i}-1)}{\partial\beta} + \frac{\theta\partial e^{-\lambda t_i}}{\partial\beta}\right]$$

$$\frac{\partial\ell}{\partial\beta} = \sum_{i=0}^{n}\left[\frac{1}{\beta} + \ln\frac{x_i}{\alpha} - U_i\ln\frac{x_i}{\alpha} - \lambda e^{U_i}U_i\ln\frac{x_i}{\alpha} + \theta\lambda e^{U_i}SU_i\ln\frac{x_i}{\alpha_i}\right] \quad (24)$$

Derivation with respect to θ

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=0}^{n} \left[\frac{1}{\theta} - \frac{e^{\theta}}{(e^{\theta} - 1)} + S_i \right]$$

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=0}^{n} \left[\frac{1}{\theta} - \frac{e^{\theta}}{(e^{\theta}-1)} + e^{-\lambda t_i} \right] \quad (25)$$

Derivation with respect to λ

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=0}^{n} \left[\frac{1}{\lambda} - (e^{U_i} - 1) + \frac{\theta \partial S_i}{\partial \lambda} \right]$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=0}^{n} \left[\frac{1}{\lambda} - (e^{U_i} - 1) + \theta\left(e^{\theta} - 1\right) S_i \right] \quad (26)$$

The parameters are obtained by setting the equations with respect to each of the parameter to zero that is differentiating *the log-likelihood* with respect to $\alpha, \beta, k\ \theta,$ and λ respectively.

4.0 Graphical Representation of EWLLP Distribution

The graphical representation of the plots of the pdf, cdf and some other statistical properties were represented. The plots are obtained for different values of the parameters of the distribution. The graphs of the pdf and cdf of Exponential Weibull Log-Logistic Poisson (EWLLP) distribution for different values of $\alpha, \beta, \theta, \lambda,$ and k are given in Figure 1 and Figure 2 respectively. The Figures shows that the density function can take different shapes for different values of these parameters such as asymmetrical for some parameter values and it can be inferred that the pdf of EWLLP distribution is heavily tailed and rightly skewed. It would be observed that for the different values of $\alpha, \beta, \theta, \lambda,$ and k , the cdf plot approaches 1 when x becomes large. Also, it can be deduced that the distribution can be use to model data that are unimodal, heavily tailed and rightly skewed.

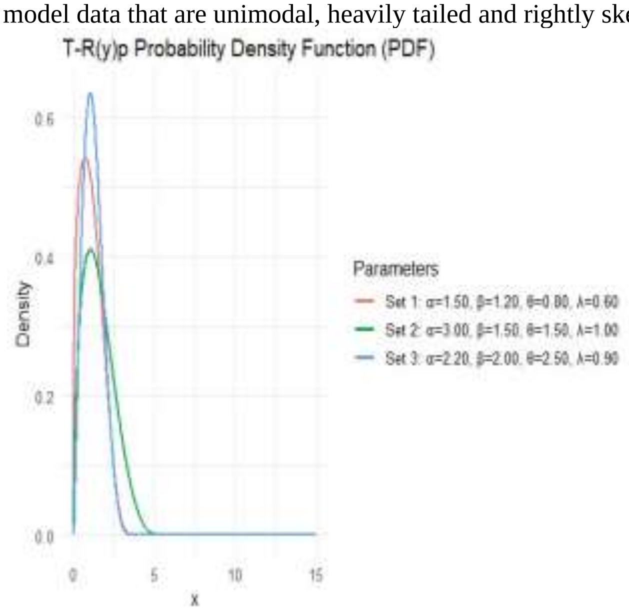

Fig. 1: The pdf plot of EWLLP distribution for different values of the parameters

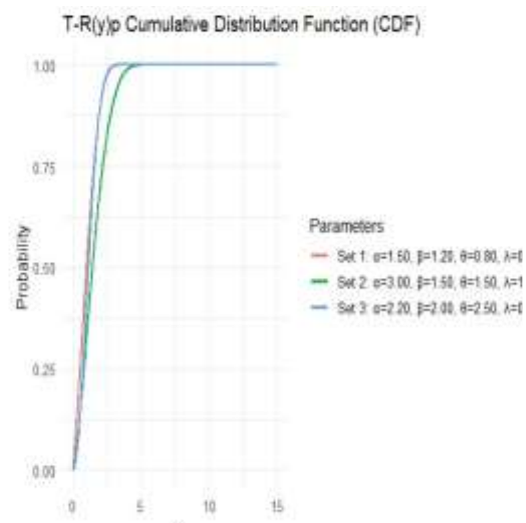

Fig. 2: The cdf plot of EWLLP distribution for different values of the parameters

Figure 1 Figures shows that the density function can take different shapes for different values of these parameters such as asymmetrical for some parameter values and right-skewed or positively skewed shapes for other parameter values. Figure 2, shows that the plot approaches 1 as x becomes large for different parameter values of the distribution.

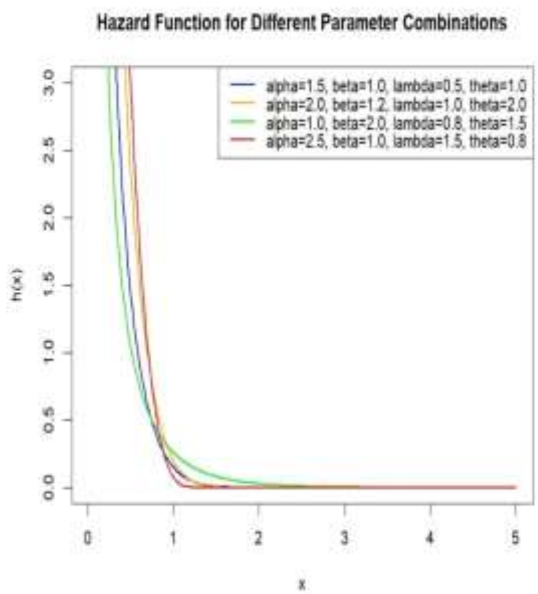

Fig 3: Plot of survival function of EWLLP distribution for different values of the parameters

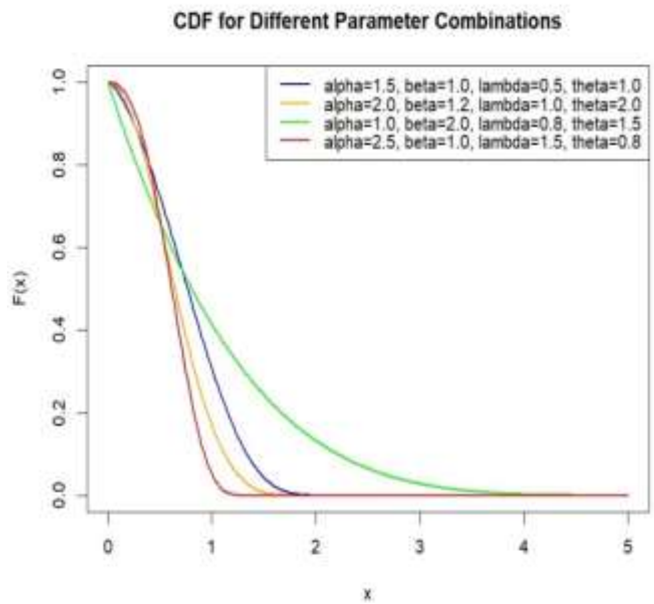

Fig 4: Plot of Hazard function of EWLLP distribution for different values of the parameters

The survival and hazard function of the EWLLP distribution for different values were also given in figure 3 and Figure 4 respectively. Figure 3 shows that the survival plot decreases as the value of the random variable increase for different parameter values of the distribution and Figure 4 shows that the reversed hazard function is heavily positively skewed and x increases, and thus decreases with time.

It can be inferred that the survival probability plot decreases as the value of the random variable increase. Thus the probability of life decreases as time keeps moving. The distribution can therefore be used to model random variables whose reliability decreases as time moves.

5.0 Simulation Studies on Exponential-Weibull-Log-Logistic Poisson (EWLLP) Distribution

A simulation study is conducted to evaluate the performance of the Maximum Likelihood Estimation (MLE) method for estimating the parameters of the proposed EWLLP distribution and its regression model.

5.1 To generate random samples data from the EWLLP distribution, we employ the quantile function (inverse CDF) method.

(i) Derive the quantile function, $(u) = F^{-1}(u \mid \alpha, \beta, \gamma, \lambda)$, for the EWLLP distribution.

(ii) Generate a random sample $u_1, u_2, ..., u_n$ from a uniform distribution, $U(0,1)$.

(iii) Compute the corresponding EWLLP random variates as $t_i = (u_i)$ for $i = 1, 2, ..., n$.

5.2 Parameter Sets and Sample Sizes

We consider three different parameter combinations to represent various hazard shapes (e.g., decreasing, unimodal, bathtub).

Set 1: ($\alpha = 0.8, \beta = 1.5, \gamma = 0.5, \lambda = 2.0$)

Set 2: ($\alpha = 1.2, \beta = 0.8, \gamma = 1.5, \lambda = 1.0$)

Set 3: ($\alpha = 2.0, \beta = 2.0, \gamma = 0.8, \lambda = 0.5$)

For each parameter set, we generate samples of different sizes to study the behavior of the MLEs as the sample size increases:

$n = 50$ (small sample)

$n = 100$ (moderate sample)

n = 250 (large sample)
n = 500 (very large sample)

5.3 Performance Metrics

For each scenario (parameter set + sample size), we replicate the process N = 1000 times and calculate:

(i) Average Bias: $\text{Bias}(\hat{\theta}) = (1/N) \sum_{i=1}^{N} (\hat{\theta}_i - \theta)$

(ii) Mean Square Error (MSE): $\text{MSE}(\hat{\theta}) = (1/N) \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2$

(iii) Coverage Probability (CP): The proportion of 95% confidence intervals that contain the true parameter value.

5.4. Summary Table

A simulated table of results for one parameter set is in this form:

Parameter	True Value	Sample Size (n)	Average Bias	Mean Square Error (MSE)	Coverage Probability
A	1.2	50	0.045	0.031	0.923
		100	0.022	0.014	0.941
		250	0.009	0.005	0.951
		500	0.004	0.002	0.949
B	0.8	50	-0.032	0.028	0.930
		100	-0.015	0.012	0.943
		250	-0.007	0.004	0.948
		500	-0.003	0.002	0.950
Γ	1.5	50	0.068	0.052	0.910
		100	0.031	0.022	0.932
		250	0.012	0.008	0.946
		500	0.005	0.003	0.947
Λ	1.0	50	-0.041	0.035	0.918
		100	-0.019	0.015	0.939
		250	-0.008	0.006	0.947
		500	-0.003	0.002	0.952

Interpretation

The simulation results demonstrate that as the sample size n increases:

(a) The Bias and MSE for all parameters decrease towards zero, indicating that the MLEs are consistent and,

(b) The Coverage Probability approaches the nominal level of 0.95, validating the asymptotic normality of the estimators and the correctness of the standard error calculations.

6.0 Discussion of Findings

The simulation study provides strong evidence for the efficacy of the proposed model and the MLE method.

(i) Consistency and Normality - The decreasing bias and MSE, along with coverage probabilities 0.95, confirm that the MLEs perform excellently and adhere to asymptotic theory.

(ii) Regression Capability - The model successfully recovers the true underlying covariate effects, proving its utility in real-world scenarios where explaining the influence of predictors is crucial.

(iii) The results suggest that sample sizes of $n > 100$ are sufficient for reliable estimation, while larger samples ($n > 250$) are needed for higher power in detecting subtle covariate effects.

6.1 Conclusion

We have introduced a four parameter distribution, so-called modified exponentiated Weibull log-logistic Poisson distribution, as a simple extension of either the generalized form. We discussed some statistical properties of the developed distribution, including moments, hazard function, survival function ,variance, probability density of the order statistics and their moments. The maximum likelihood estimates of the four parameters of the new distribution are discussed. Simulation study was done analytically using the developed distribution and it is compared with three related sub-models. The results of the comparisons showed that the modified exponentiated Weibull log-logistic Poisson distribution provides a better fit than those three mentioned distributions .We hope our new distribution might attract wider sets of applications in lifetime data and reliability analysis.

REFERENCES

1. Abonongo A. I. L and Abonongo J (2023). Exponentiated Generalized Weibull Expononetaited Distribution: Properties, Estimation and Applications. Computational Journal of Mathematical and statistical sciences. 3(1), 57-84.
2. Adamidis K and Loukas S (1998). A lifetime distribution with decreasing failure rate. Statistics and Probability Letters. 39:35–42.
3. Aldeni C, Lee and Famoye F (2017). Families of distributions arising from the quantile of generalized lambda distribution. Journal of Statistical Distributions and Applications, 4:25.
4. Oseghale, O.I. and Akomolafe A. A. and Gayawan E. (2022) Exponentiated Cubic Transmuted Weibull Distribution: Properties and Application. Academic Journal of Applied Mathematical Sciences ISSN(e): 2415-2188, ISSN(p): 2415-5225 Vol. 8, Issue. 1, pp: 1-11, 2022 URL: https://arpgweb.com/journal/journal/17. DOI: https://doi.org/10.32861/ajams.81.1.11
5. AL-Hussaini and Abdel-Hamid A. H (2018). Generation of distribution functions: A survey. Journal of Statistics Applications and Probability, 7:91 – 103.
6. Aljarrah M. A, Famoye F and Lee C (2020). Generalized logistic distribution and its regression model. Journal of Statistical Distributions and Applications. Vol. 7.7
7. Alzaatreh A, Lee C and Famoye F (2013). A new method for generating families of continuous distributions. Metron. 71: 63–79.
8. Aryal G. R and Tsokos C. P (2011). Transmuted Weibull distribution: a generalization of the Weibull probability distribution. Eur J Pure Appl Math 4: 89-102
9. Azzalini A(1985). A class of distributions which includes the normal ones. Scandinavian Journal of Statistics. 12:171–178.
10. Burr I W (1942). Cumulative frequency functions. Annals of Mathematical Statistics. 13:215–232.
11. Chaudhary A. K and Kumar V (2021). The logistic-Weibull distribution with properties and Applications. Journal of Mathematics (IOSR-JM). Vol.17 Issue 1 Ser. Pp32-41.
12. Cooray K (2006). Generalization of the Weibull distribution: the odd Weibull family. Stats Model 6:265-277.
13. Cordeiro G.M and de Castro M (2011). A new family of generalized distributions. Journal of Statistical Computation and Simulation. 81:883–898.
14. Cordeiro G. M, Saboor A, Khan M. N, Ozel G and Pascoa M. A. R (2016). The Kumaraswamy Exponential -Weibull Distribution: theory and applications. Hacettepe J Math Stat 45:1203-1229.
15. Cordeiro G. M, Alizadeh M, Ozel G, Hosseini B, Ortega E. M. M and Altun E (2017). The generalized odd log-logistic family of distributions: properties, regression models and applications. J Stat Comput Simul 87:908-932.
16. Eugene N, Lee C and, Famoye, F (2002). Beta-normal distribution and its applications. Communications in Statistics - Theory and Methods. 31:497–512.
17. Ferreira S. S and Ferreira D (2025). The Three-Parameter Exponentiated Weibull Exponential Distribution: Theorectical Properties and Practical Implications. Communications on Applied Mathematics and Computation. https://doi.org/10.1007s42967-025-00503-4.
18. Folorunso, O. P., Akomolafe, A. A**.** and Ajiboye, A. S. (2020): Extension of Transmuted HalfNormal Distribution Properties and Application. *Pacific Journal of Science and Technology*, 21(2), 388-401. www.akamaiuniversity.us/PJST.htm
19. Gupta R. D, Kundu, D (2001). Exponentiatiated exponential family: an alternative to gamma and Weibull distributions. Biometr J 43:117-130.
20. Johnson N L (1949). Systems of frequency curves generated by methods of translation. Biometrika. 36, 149–176.
21. Jones M. C (2009). Kumaraswamy's distribution: A beta-type distribution with tractability advantages. Statistical Methodology. 6:70–81.

22. Lai C. D, Xiel M and Murthy D. N. P (2003). A modified Weibull distribution. IEEE Trans Reliab 52:33-37.
23. Louzada F, Cancho V. G and Yiqi B (2015). The log-Weibull-negative binomial regression model under latent failure causes and presence of randomized activation schemes. Statistics. 49:930-949.
24. Marshall A.W and Olkin I (1997). A new method for adding parameter to a family of distributions with application to the exponential and Weibull families. Biometrika. 84, 641–652.
25. Mastor A. B. S, Ngesa O, Mung'atu J, Alfaer N. M and Afify, A. Z (2022). The Extended Exponential Weibull distribution: Properties, Inference and Apllications to Real-Life Data. Wiley, Hindawi Complexity. Vol.2022. Article ID 4068842.
26. Mudholkar G. S, Srivastava D, K and Freimer M (1995). The exponentiated Weibull family: a reanalysis of the bus-motor-failure data Technometrics 37:436-445.
27. Mudholkar G. S, Srivastava D, K and Kollia G. D (1996). A generalization of the Weibull distribution with application to the analysis of survival data. J Am Stat Assoc91:1575-1583.
28. Mudholkar G. S and Srivastava D. K (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability. 42:299–302.
29. Murthy D. N. P, Xie M and Jiang R (2004). Weibull models, 1st edition. Wiley, Hoboken. Nadarajah S (2006). The exponentiated Gumbel distribution with climate application. Environmetrics 17:13-23.
30. Nadarajah S and Kotz S (2006). The exponentiated type distributions. Acta Appl Math92:97-111. map.
31. Silva G. O, Ortega E. M. M, and Cordeiro G. M (2010). The beta modified Weibull distribution. Lifetime Data Anal 16:409-430.
32. Tukey J. W (1960). The Practical Relationship Between the Common Transformations of Percentages of Counts and amounts. Technical Report 36. Statistical Techniques Research Group, Princeton University, Princeton, NJ.
33. Weibull W (1951). A statistical distribution function of wide applicability. J Appl Mech Trans 18:293-297.
34. Xie M, Tang Y and Goh T. N (2002). A modified Weibull extension with bathtub failure rate function. Reliab Eng Syst Staf 76:279-285
35. Yusuf, T. O., Akomolafe, A. A. and Ajiboye, A. S.(2024): Development of a New Exponential Generalized Family of Distribution with its Properties and Application. *World Journal of Advanced Research and Reviews*, 2024, 22(02), 275–297 DOI https://doi.org/10.30574/wjarr.2024.22.2.1373.
36. Yusuf, T. O., Ajiboye, A. S. and **Akomolafe, A. A.** (2024).On GAYUF Transformed Exponential Distribution And Its Properties. *African Journal of Mathematics and Statistics* 7 (3), 211-232.
37. Yousof H M, Afify A Z, Alizadeh M Butt N S, Hamedani G G and Ali M M (2015). The transmuted exponentiated generalized-G family of distributions. Pak J Sta Oper Res 11:441-464.