Spectral Relationships of Some Mixed Integral Equations of the First Kind

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Abstract: Here, the existence of a unique solution of mixed integral equation (**MIE**) of the first kind in three dimensions is discussed in the space $L_2[\Omega]\times C[0,T]$, $T<1$; Ω is the domain of integration with respect to position. A numerical method is used to obtain system of Fredholm integral equations (**SFIEs**). Many spectral relationships (**SRs**), when the kernel of position takes a logarithmic form, Carleman function, elliptic kernel, potential function and generalized potential function are obtained in this work. In addition, many important new and special cases are considered and discussed.

[S. J. Monaquel. **Spectral Relationships of Some Mixed Integral Equations of the First Kind**. J Am Sci. 2012;8(5): 89- 99]. (ISSN: 1545-1003). http://www.americanscience.org. 13

Keywords: Mixed integral equation, system of Fredholm integral equations, generalized kernel, Weber-Sonien integral formula, spectral relationships.
MSC: 45B05, 45R10.

1. Introduction

The mathematical formulation of physical phenomena, population genetics, mechanics and contact problems in the theory of elasticity, often involves singular integral equation with different kernels. The monographs [1-6] contain many different **SRs** for different kinds of integral equations, in one, two and three dimensional. In addition, in [7, 8] using Krein's method, Mkhitarian and Abdou obtained many **SRs** for the **FIE** of the first kind with logarithmic kernel and Carleman function, respectively.
Consider the **MIE**

$$\lambda_1\int_\Omega k\left|\frac{x-y}{\lambda}\right|\Phi(y,t)\,dy+\lambda_2\int_0^t\int_\Omega F(|t-\tau|)\,k\left|\frac{x-y}{\lambda}\right|\Phi(y,\tau)\,dy\,d\tau=f(x,t)$$

$$\left(x=\bar{x}\left(x_1,x_2,x_3\right),\ y=\bar{y}\left(y_1,y_2,y_3\right)\right)\qquad(1.1)$$

under the condition

$$\int_\Omega \Phi(x,t)\,dx=P(t)\qquad(1.2)$$

The integral equation (1.1), under the condition (1.2), can be investigated from the mixed contact problem of a rigid surface (G,v), G is the displacement magnitude and v is the Poisson's coefficient, having an elastic material occupying the domain Ω, where Ω is the domain of integration with respect to position, through the time t; $t\in[0,T]$, $T<1$. The given function $f(x,t)$ is the sum of two functions, the first function $\delta(t)$ represents the displacement of the surface, under the action of the pressure of (1.2) $P(t)$, $t\in[0,T]$, $T<1$, and the second function $f_1(x)$ describes the basic formula of the surface. Here, λ, λ_1 and λ_2 are constants, may be complex, and having many physical meanings. The unknown function $\Phi(x,t)$, represents the normal stresses between the layers of the two surfaces. The known function $k\left|\frac{x-y}{\lambda}\right|$ is the kernel of position and has a singular term, while $F(|t-\tau|)$ is the kernel of Volterra integral term in time, and represents the resistance of the layer of the surface against the pressure $P(t)$.

In order to guarantee the existence of a unique solution of (1.1), we assume the following conditions:

(i) The kernel of the position $k\left|\frac{x-y}{\lambda}\right|$, $x=\bar{x}(x_1,x_2,x_3)$ and $y=\bar{y}(y_1,y_2,y_3)$, satisfies in $L_2(\Omega)$, the condition

$$\left\{ \int_{\Omega}\int_{\Omega} k^2\left|\frac{x-y}{\lambda}\right| dx\ dy \right\}^{\frac{1}{2}} = A\ , \quad (\text{ A: constant })$$

(ii) The positive continuous function $F(|t-\tau|)\in C([0,T]\times[0,T])$, and satisfies $F|t-\tau| < B$, B is a constant, for all values $(t,\tau)\in[0,T]$, $T<1$.

(iii) The given function $f(x,t)$ with its first partial derivatives are continuous and belong to the class $L_2(\Omega)\times C[0,T]$, where

$$\| f \|_{L_2\times C} = \max_{0\le t\le T} \int_0^t \{ f^2(x,\tau)\ dx \}^2\ d\tau\ ,$$

(iv) The unknown function $\Phi(x,t)$ satisfies Hölder condition with respect to time and Lipschitz condition with respect to position.

In this work, a numerical method is used to transform the **MIE** (1.1) into **SFIEs** of the first kind. In addition, the potential theory method, Fourier transformation method, orthogonal polynomial method and Krein's method will be used to establish many theorems for obtaining the **SRs** of the **SFIEs** (2.1), under the condition (2.3), in one, two and three dimensional in the space $L_2[\Omega]\times C[0,T]$, $T<1$; Ω is the domain of integration with respect to position. The kernel of position of (2.1) will take the following forms: logarithmic form, Carleman function, elliptic and potential kernels, and generalized potential kernel. Moreover, many important new cases will be discussed here.

2. System of Fredholm integral equations

If we divide the interval [0,T], 0 ≤ t ≤ T <1as $0\le t_0<t_1<\ldots<t_i=T$, when $t=t_\ell$, $\ell=0,1,2,\ldots,i$, the **MIE** (1.1) takes the form

$$\lambda_2\sum_{j=0}^{\ell}u_j F_{j,\ell}\int_{\Omega}k\left|\frac{x-y}{\lambda}\right|\Phi_j(y)\ dy+\lambda\int_{\Omega}k\left|\frac{x-y}{\lambda}\right|\Phi_\ell(y)\ dy+O(h_\ell^{p+1})=f_\ell(x)$$

$$(h_\ell\to 0,\ p>0) \qquad (2.1)$$

where $h_\ell=\max_{0\le j\le\ell}h_j$ and $h_j=t_{j+1}-t_j$,

Here, we used the following notations

$$F(|t_\ell-t_j|)=F_{\ell,j}\ ,\quad \Phi(y,t_\ell)=\Phi_\ell(y)\ ,\quad f(x,t_\ell)=f_\ell(x) \qquad (2.2)$$

The values u_j and the constant p depend on the number of derivatives of $F(|t-\tau|)$ with respect to t, see [9, 10]. Also, the boundary condition (1.2) becomes

$$\int_{\Omega}\varphi_\ell(x)\ dx = P_\ell\ ,\quad \ell=0,1,2,\ldots,N \qquad (2.3)$$

The formula (2.1) represents **SIEs** of the first kind, where it's solution depends on the kind of the kernel $k\left|\frac{x-y}{\lambda}\right|$ and the domain of integration Ω. In the next applications we will neglect the error term $O(h_\ell^{p+1})$.

3. Theorems of spectral relationships

In this section, we obtain the **SRs** of the **SFIEs** in one, two and three dimensional using different domains and suitable methods.

3.1 SIEs with logarithmic kernel

Consider **SFIEs** of the first kind,

$$\lambda_2\sum_{j=0}^{\ell}u_j F_{j,\ell}\int_{-1}^{1}k\left(\frac{|x-y|}{\lambda}\right)\Phi_j(y)dy+\lambda\int_{-1}^{1}k\left(\frac{|x-y|}{\lambda}\right)\Phi_\ell(y)dy=f_\ell(x) \qquad (3.1)$$

$$k(z)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\tanh u}{u}\exp(iuz)\,du\ ,\quad z=\frac{x-y}{\lambda}$$

under the conditions

$$\int_{-1}^{1}\varphi_\ell(x)\ dx = P_\ell \qquad (3.2)$$

The kernel of **SFIEs** (3.1) can be written in the form (see [11])

$$k(z)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\tanh u}{u}\exp(iuz)\,du=-\ln\left|\tanh\frac{\pi z}{4}\right|,\ z=\frac{x-y}{\lambda} \qquad (3.3)$$

If $\lambda \to \infty$, and z is very small, so that $\tanh z \cong z$, then we may write

$$\ln\left|\tanh\frac{\pi z}{4}\right| \cong \ln|x-y| - d \quad , \quad d = \ln\frac{4\lambda}{\pi} \tag{3.4}$$

Hence, we have

$$\lambda_2 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-1}^{1} [\ln|x-y| - d] \Phi_j(y)\, dy + \lambda_1 \int_{-1}^{1} [\ln|x-y| - d] \Phi_\ell(y)\, dy = f_\ell(x) \tag{3.5}$$

Let $T_n(x) = \cos(n\cos^{-1}x),\ x \in [-1,1],\ n \geq 0$ denotes the Chebyshev polynomials of the first kind, while $U_n(x) = \dfrac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)},\ n \geq 0$, denotes the Chebyshev polynomials of the second kind. It is well known that $\{T_n(x)\}$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$, while $\{U_n(x)\}$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{\frac{1}{2}}$. It appears reasonable to attempt a series expansion to $\Phi_\ell(x)$ in Eq. (3.1) in terms of Chebyshev polynomials of the first kind. This choice is not arbitrary since one can identity a portion of the integral as the weight function associated with $T_n(x)$.

For convenience, we use the orthogonal polynomials method with some well known algebraic and integral relations associated with Chebyshev polynomials see [12,13]. Thus, in this aim, we represent $\Phi_\ell(x)$, $f_\ell(x)$, in the following forms

$$\Phi_\ell(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n_\ell=0}^{\infty} a_{n_\ell} T_{n_\ell}(x), \quad f_\ell(x) = \sum \frac{f_{n_\ell} T_{n_\ell}}{\sqrt{1-x^2}} \tag{3.6}$$

Using the above expressions of (3.6) in (3.5) we have the following:

Theorem 1: The **SRs** of the **MIE** (3.1), under the conditions (3.2), when the kernel takes a logarithmic function are given as :

$$\lambda_2 \sum_{J=0}^{\ell} u_j F_{j,\ell} \int_{-1}^{1} \left[\ln\frac{1}{|x-y|} + d\right] \frac{T_{n_j}(y)\, dy}{\sqrt{1-y^2}} + \lambda_1 \int_{-1}^{1} \left[\ln\frac{1}{|x-y|} + d\right] \frac{T_{n_\ell}(y)\, dy}{\sqrt{1-y^2}}$$

$$= \begin{cases} \pi(\ln 2 + d)\left(\lambda_2 \sum_{j=0}^{\ell} u_j F_{j,\ell} + \lambda_1\right) & n_j = 0 \\ \pi\lambda_2 \left(\sum_{j=0}^{\ell} \dfrac{u_j F_{j,\ell} T_{n_j}(x)}{n_j} + \pi\lambda_1 \dfrac{T_{n_\ell}(x)}{n_\ell}\right) & n_j \geq 1. \end{cases} \tag{3.7}$$

Different new cases can be established from (3.7) as the following:

(1) Differentiating (3.7) with respect to x, we get

$$\lambda_2 \sum_{J=0}^{\ell} u_j F_{j,\ell} \int_{-1}^{1} \frac{T_{n_j}(y)\, dy}{(y-x)\sqrt{1-y^2}} + \lambda_1 \int_{-1}^{1} \frac{T_{n_\ell}(y)\, dy}{(y-x)\sqrt{1-y^2}}$$

$$= \pi\lambda_2 \sum_{j=0}^{\ell} u_j F_{j,\ell} U_{n_j-1}(x) + \pi\lambda_1 U_{n_\ell-1}(x) \qquad n_j \geq 1, \tag{3.8}$$

Hence we get:

$$\lambda_2 \sum_{J=0}^{\ell} u_j F_{j,\ell} \int_{-1}^{1} \frac{T_{n_j}(y)\, dy}{(y-x)\sqrt{1-y^2}} + \lambda_1 \int_{-1}^{1} \frac{T_{n_\ell}(y)\, dy}{(y-x)\sqrt{1-y^2}} = 0 \tag{3.9}$$

Thus, the result of (3.8) leads to the **SRs** of the **SFIEs** with Cauchy kernel. While (3.9) leads, directly to the fact

$$\int_{-1}^{1} \frac{dy}{(y-x)\sqrt{1-y^2}} = 0 .$$

(2) If $n_\ell = 2m_\ell$, $x = \dfrac{\sin\xi/2}{\sin\alpha/2}$, $y = \dfrac{\sin\eta/2}{\sin\alpha/2}$, $(-\alpha \leq \xi, \eta \leq \alpha,\ \alpha = \pi)$, in (3.7), we have the following **SRs**

$$\lambda_2 \sum_{J=0}^{\ell} u_j\, F_{j,\ell} \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] \frac{T_{2m_j}\left(\frac{\sin\eta/2}{\sin\alpha/2}\right)\cos(\eta/2)\,d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}}$$

$$+\lambda_1 \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] \frac{T_{2m_j}\left(\frac{\sin\eta/2}{\sin\alpha/2}\right)\cos(\eta/2)\,d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}}$$

$$= \begin{cases} \pi\,\lambda_2\left[\ln\left(\frac{2}{\sin\alpha/2}\right)+d\right]\sum_{J=0}^{\ell} u_j F_{j,\ell} + \pi\,\lambda_1\left[\ln\left(\frac{2}{\sin\alpha/2}\right)+d\right] & n_\ell = 0 \\[2ex] \pi\,\lambda_2 \sum_{j=0}^{\ell} \frac{u_j F_{j,\ell} T_{2m_j}\left(\frac{\sin\xi/2}{\sin\alpha/2}\right)}{2m_j} + \pi\,\lambda_1 \frac{T_{2m_\ell}\left(\frac{\sin\xi/2}{\sin\alpha/2}\right)}{2m_\ell} & n_j \ge 1. \end{cases} \tag{3.10}$$

(3) If $n_\ell = 2m_\ell + 1$, $x = \frac{\tan\xi/2}{\tan\alpha/2}$, $y = \frac{\tan\eta/2}{\tan\alpha/2}$ $(-\alpha \le \xi, \quad \eta \le \alpha, \quad \alpha = \pi)$, Eq. (3.7) yields

$$\lambda_2 \sum_{J=0}^{\ell} u_j\, F_{j,\ell} \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] T_{2m_j+1}\left(\frac{\tan\eta/2}{\tan\alpha/2}\right) \frac{\cos(\eta/2)\;d\eta}{\sqrt{2\,(\cos\eta-\cos\alpha)}} +$$

$$+\lambda_1 \int_{-\alpha}^{\alpha} \left[\ln \frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|} + d \right] T_{2m_\ell+1}\left(\frac{\tan\eta/2}{\tan\alpha/2}\right) \frac{\cos(\eta/2)\,d\eta}{\sqrt{2\,(\cos\eta-\cos\alpha)}}$$

$$= \pi\,\lambda_2 \sum_{j=0}^{\ell} \frac{u_j F_{j,\ell}\; T_{2m_j+1}\left(\frac{\tan\xi/2}{\tan\alpha/2}\right)}{2m_j+1} + \pi\,\lambda_1 \frac{T_{2m_\ell+1}\left(\frac{\tan\xi/2}{\tan\alpha/2}\right)}{2m_\ell+1} \tag{3.11}$$

(4) Using the following relations

$$\cos\left(\frac{\xi}{2}\right)\cos\left(\frac{\eta}{2}\right) = \cos\left(\frac{\xi-\eta}{2}+\frac{\eta}{2}\right)\cos\left(\frac{\eta}{2}\right) = \cos\left(\frac{\eta}{2}\right)\cdot\cos\left(\frac{\eta-\xi}{2}+\frac{\eta}{2}\right) \tag{3.12}$$

the formula (4.8) leads to the following **SRs**

$$\lambda_2 \sum_{j=0}^{\ell} u_j\, F_{j,\ell} \int_{-\alpha}^{\alpha} \frac{\cot\left(\frac{\eta-\xi}{2}\right) T_{n_j}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right)\cos\left(\frac{\eta}{2}\right)d\eta}{\sqrt{2\,(\cos\eta-\cos\alpha)}} + \lambda_1 \int_{-\alpha}^{\alpha} \frac{\cot\left(\frac{\eta-\xi}{2}\right) T_{n_\ell}\left(\frac{\tan\frac{\eta}{2}}{\tan\frac{\alpha}{2}}\right)\cos\left(\frac{\eta}{2}\right)d\eta}{\sqrt{2\,(\cos\eta-\cos\alpha)}}$$

$$
=\begin{cases} 0 & n_j=0 \\[2ex] 2\,\lambda_2\sum_{j=0}^{\ell} u_j F_{j,\,\ell}\ \csc\left(\frac{\alpha}{2}\right)\ U_{2m_j-1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right)+2\,\lambda_1\csc\left(\frac{\alpha}{2}\right)U_{2m_\ell-1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right) & (\ n_k=2m_k,\ \ m=1,\ 2,\dots\ ) \\[2ex] 2\,\lambda_2\sum_{j=0}^{\ell} u_j F_{j,\,\ell}\left[\csc\left(\frac{\alpha}{2}\right)\ U_{2m_j-1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right)+(-1)^2\frac{\sin\alpha}{1+\cos\alpha}\left[\tan\frac{\alpha}{4}\right]^{2m_j-2}\right] & \\ +\,2\,\lambda_1\left[\cos\left(\frac{\alpha}{2}\right)U_{2m_\ell-1}\left(\frac{\tan\frac{\xi}{2}}{\tan\frac{\alpha}{2}}\right)+(-1)^2\frac{\sin\alpha}{1+\cos\alpha}\left[\tan\frac{\alpha}{4}\right]^{2m_\ell-2}\right] & n_k=2m_k-1 \end{cases} \tag{3.13}
$$

Here, in (3.13) we obtain the **SRs** of the integral operator with Hilbert kernel for different values of n_ℓ *and* n_j, $j=0, 1, 2,\dots, \ell$.

3.2 SIEs with Carleman function

The importance of Carleman function came from the work of Arutiunion [14], who has shown that, the contact problem of the nonlinear theory of plasticity, in its first approximation reduce to **FIE** of the first kind with Carleman function. If we consider, in the formula (3.1) the following singular kernel $k\left|\frac{x-y}{\lambda}\right|=\left|x-y\right|^{-\nu},\ 0\le\nu<1;\ \Omega\in[\,-1,1\,]$, we have the following **SFIEs:**

$$
\lambda_2\sum_{j=0}^{\ell}u_j F_{j,\ell}\int_{-1}^{1}\left|x-y\right|^{-\nu}\Phi_j(y)\,dy+\lambda_1\int_{-1}^{1}\left|x-y\right|^{-\nu}\Phi_\ell(y)\,dy=f_\ell(x) \tag{3.14}
$$

To obtain the solution of the formula (3.14), we assume and represent the unknown and known functions, respectively in the following form:

$$
\Phi_k(x)=\frac{1}{\left(1-x^2\right)^{\left(\frac{1-\nu}{2}\right)}}\sum_{n=0}^{\infty}a_{nk}\ C_{2n_k}^{\frac{\nu}{2}}(x),
$$

$$
f_\ell(x)=\frac{1}{\left(1-x^2\right)^{\left(\frac{1-\nu}{2}\right)}}\sum_{n=0}^{\infty}f_{n\ell}\ C_{2n_k}^{\frac{\nu}{2}}(x). \tag{3.15}
$$

Here, $C_{2n}^{\nu}(x)$ are Gegenbauer polynomials, a_{nk} are the unknown coefficients and $f_{n\ell}$ are the known coefficients. Using the potential theory method [3], and the following relations [12]

1. $n\,C_n^{\nu}(x)=2\,\nu\left[x\ C_{n-1}^{\nu+1}(x)-C_{n-2}^{\nu+1}(x)\right],$

2. $\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,C_n^{\nu}(x)\,dx=$

$$
\frac{2^{\alpha+\beta+1}\,\Gamma(1+\alpha)\,\Gamma(1+\beta)\,\Gamma(n+2\nu)}{n!\,\Gamma(2\nu)\,\Gamma(\alpha+\beta+2)}\times{}_3F_2\left(-n,\,n+2\nu,\,\alpha+1;\,\nu+\tfrac{1}{2},\,\alpha+\beta+2;\,1\right)
$$

$$
\int_{-1}^{1}\left(1-x^2\right)^{\frac{1}{2}\nu-1}C_{2n}^{\nu}(x)\,dx=\frac{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)}\,C_n^{\frac{\nu}{2}}\left(2x^2-1\right),\ \ (\operatorname{Re}\nu>0)
$$

3. $\int_{-1}^{1}\left(1-x^2\right)^{\nu-\frac{1}{2}}\left[C_n^{\nu}(x)\right]^2dx=\dfrac{\pi\,2^{1-2\nu}\,\Gamma(2\nu+n)}{n!\,(n+\nu)\,[\Gamma(\nu)]^2},\ \ \operatorname{Re}\nu>-\dfrac{1}{2}$

where $\Gamma(x)$ is the Gamma function and ${}_3F_2(.,.,.;.;.;1)$ is the Generalized hypergeometric function, we have the following:

Theorem 2: The **SRs** of the **MIE** (3.1), under the condition (3.2), when the kernel takes the Carleman function form are

$$\lambda_2 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-1}^{1} \frac{C_{2n_j}^{\frac{v}{2}}(y)\,dy}{|x-y|^{v}(1-y^2)^{\frac{1-v}{2}}} + \lambda_1 \int_{-1}^{1} \frac{C_{2n_\ell}^{\frac{v}{2}}(y)\,dy}{|x-y|^{v}(1-y^2)^{\frac{1-v}{2}}}$$

$$= \lambda_2 \sum_{j=0}^{\ell} u_j F_{j,\ell}\, \mu_{2n_j} C_{2n_j}^{\frac{v}{2}}(x) + \lambda_1 \mu_{2n_\ell} C_{2n_\ell}^{\frac{v}{2}}(x) \quad , \quad n_j \geq 0 \tag{3.16}$$

and

$$\lambda_2 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-1}^{1} \frac{C_{2n_j-1}^{\frac{v}{2}}(y)\,dy}{|x-y|^{v}(1-y^2)^{\frac{1-v}{2}}} + \lambda_1 \int_{-1}^{1} \frac{C_{2n_\ell-1}^{\frac{v}{2}}(y)\,dy}{|x-y|^{v}(1-y^2)^{\frac{1-v}{2}}}$$

$$= \lambda_2 \sum_{j=0}^{\ell} u_j F_{j,\ell}\, \mu_{2n_j-1} C_{2n_j-1}^{\frac{v}{2}}(x) + \lambda_1 \mu_{2n_\ell-1} C_{2n_\ell-1}^{\frac{v}{2}}(x) \quad , \quad n_j \geq 1 \tag{3.17}$$

where

$$\mu_{2n_k} = \pi\, \Gamma(2n_k + v)\left[\Gamma(2n_k+1)\,\Gamma(v)\cos\left(\frac{\pi v}{2}\right)\right]^{-1} \quad (n_k \geq 0)$$

3.3. *SIEs with potential kernel in finite domain*

Assume the domain of integration Ω, in (2.1), in the form $\Omega = \{(x,y,z) \in \Omega : \sqrt{x^2+y^2} \leq a,\ z=0\}$ and the kernel takes the potential function form $k(x-\xi,\ y-\eta) = [(x-\xi)^2 + (y-\eta)^2]^{-\frac{1}{2}}$. Hence, we have the **SFIEs** with potential kernel. Using the polar coordinates and then, using the separation of variables, the **SIFs** (2.1), yields

$$\lambda_2 \sum_{j=0}^{\ell} u_j F_{j,k} \int_0^a \rho L_m(r,\rho)\Phi_j^{(m)}(\rho)\,d\rho + \lambda_1 \int_0^a \rho L_m(r,\rho)\Phi_\ell^{(m)}(\rho)\,d\rho = f_\ell^{(m)}(r) \tag{3.18}$$

where

$$L_m(r,\rho) = \int_{-\pi}^{\pi} \frac{\cos m\psi\ d\psi}{\sqrt{r^2+\rho^2-2r\rho\cos\theta}}$$

Using the following relations, see [16]

$$\int_0^{2\pi} \frac{\cos m\psi\ d\psi}{[1-2z\cos\psi+z^2]^{\alpha}} = \frac{2\pi(\alpha)_m z^m}{m!} F(\alpha, m+\alpha, m+1, z^2),$$

$$|z|<1,\ \ Re\,\alpha>0\ ,\ (\alpha)_m = \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)}$$

and

$$F\left(\alpha, \alpha+\frac{1}{2}-\beta, \beta+\frac{1}{2}, z^2\right) = (1+z)^{-2\alpha} F\left(\alpha,\ \beta;\ 2\beta;\ \frac{4z}{1+z^2}\right) \times$$

$$\int_0^{\infty} J_\alpha(ax)\,J_\alpha(bx)\,x^{-\beta}\,dx = \frac{2^{-\beta}a^{\beta}b^{\alpha}\Gamma\left(\alpha+\frac{1-\beta}{2}\right)}{(a+b)^{2\alpha-\beta+1}\Gamma(1+\alpha)\,\Gamma\left(\frac{1+\beta}{2}\right)} \times F\left(\alpha+\frac{1-\beta}{2},\ \alpha+\frac{1}{2};\ 2\alpha+1;\ \frac{4ab}{(a+b)^2}\right)$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function, $\Gamma(x)$ is the Gamma function and $J_n(x)$ is the Bessel function, the **SFIEs** (3.18) takes the form

$$\lambda_2 \sum_{j=0}^{\ell} u_j F_{j,k} \int_0^a K_m(r,\rho)\, Z_j^{(m)}(\rho)\, d\rho + \lambda_1 \int_0^a K_m(r,\rho)\, Z_\ell^{(m)}(\rho)\, d\rho = g_\ell^{(m)}(r)$$

$$\left(Z_j^m(r) = \sqrt{r}\,\Phi_j^{(m)} \quad , \quad g_\ell^{(m)} = \sqrt{r}\, f_\ell^{(m)}(r)\right) \tag{3.19}$$

where

$$K_m(r,\rho) = 2\pi\sqrt{r\rho}\int_0^{\infty} J_m(u\rho)J_m(ur)\,du \tag{3.20}$$

Eq. (3.19) represents **SFIEs** of the first kind with kernel (3.20) takes a form of Weber-Sonien integral formula.

Assume the solution of (3.19), at $a=1$, in the form

$$Z_k^{(m)}(r) = \frac{1}{\sqrt{1-r^2}} \sum_{n_k=0}^{\infty} a_{n_k}^{(m)} P_{2n_k}^{(m)}\left(\sqrt{1-r^2}\right),$$

$$(k=0,1,2,\ldots,\ell) \tag{3.21}$$

where $P_{2n}(y)$ is the Legendre polynomial. Then, using potential theory method [3] and orthogonal polynomials method [15], we obtain the following:

Theorem 3: The **SRs** of the **SFIEs** (3.1), under the condition (3.2), when the kernel takes a potential function form (3.20) are

$$\lambda_2 \sum_{j=0}^{\ell} u_j\, F_{j,\ell} \int_0^1 \frac{K_m(r,\rho) P_{m_j}^{(n,-\frac{1}{2})}\left(1-2\rho^2\right) d\rho}{\sqrt{1-\rho^2}} + \lambda_1 \int_0^1 \frac{K_m(r,\rho) P_{m_\ell}^{(n,-\frac{1}{2})}\left(1-2\rho^2\right) d\rho}{\sqrt{1-\rho^2}}$$

$$= \lambda_2\, r^n \sum_{j=0}^{\ell} \mu_{m_j}\, u_j\, F_{j,\ell}\, P_{m_j}^{(n,-\frac{1}{2})}(1-2r^2) + \lambda_1\, r^n\, \mu_{m_\ell}\, P_{m_\ell}^{(n,-\frac{1}{2})} \tag{3.22}$$

where, in general, $u_{m_k} = \dfrac{\Gamma^2\left(\frac{1}{2}+m_k\right)}{(2m_k)!\,\Gamma(1+n\,m_k)}$, and $P_m^{(\alpha,\beta)}(x)$ is a Jacobi polynomial.

3.4 SIEs with generalized potential kernel in finite domain

When the modules of the elasticity of the contact problem is changing according to $\sigma_i = K_0\, \varepsilon_i^{v},\ \ 0 \le v < 1$, where σ_i and ε_i are the stress and strain rate intensities, respectively, while K_0 and $\mathbf{V}$ are the physical constants, see [5]. For this, the kernel of Eq. (2.1) takes the form

$$K(x-\xi,\, y-\eta) = \left[(x-\xi)^2 + (y-\eta)^2\right]^{-v} \qquad 0 \le v < 1 \tag{3.23}$$

The kernel of Eq. (3.23) is called the generalized potential kernel.
Using (3.23) in (2.1) where, $\Omega = \left\{(x,y,z) \in \Omega : \sqrt{x^2+y^2} \le a,\ z=0\right\}$,we can arrive to the following **SFIEs**.

$$\lambda_2 \sum_{j=0}^{\ell} u_j\, F_{j,k} \int_0^1 K_m^{(v)}(r,\rho)\, Z_j^{(m)}(\rho)\, d\rho + \lambda_1 \int_0^1 K_m^{(v)}(r,\rho)\, Z_\ell^{(m)}(\rho)\, d\rho = g_\ell^{(m)}(r), \tag{3.24}$$

where

$$K_m^{(v)}(r,\rho) = c\sqrt{r\,\rho} \int_0^\infty u^{2v-1} J_m(u\,\rho) J_m(u\,r)\, du \quad , \quad \left(c = \frac{\pi\,\Gamma(1-v)\cdot 2^{2(1-v)}}{\Gamma(v)}\right) \tag{3.25}$$

The kernel of (3.25) takes a generalized form of Weber-Sonien integral formula.
Representing the unknown functions $Z_j^{(m)}$ and the known functions $g_\ell^{(m)}(r)$, respectively in the Jacobi polynomials form

$$Z_k^{(m)}(r) = \frac{1}{(1-r^2)^{\sigma}} \sum_{n_k=0}^{\infty} a_{n_k}^{(m)} \cdot {}^{(m)}P_{n_k}^{(m,-\sigma)}\left(1-2r^2\right),$$

$$g_k^{(m)}(r) = \frac{1}{(1-r^2)^{\sigma}} \sum_{n_k=0}^{\infty} g_{n_k}^{(m)} \cdot {}^{(m)}P_{n_k}^{(m,-\sigma)}\left(1-2r^2\right) \tag{3.26}$$

Then, using Krein's method, see [5], we can obtain the following:

Theorem 4: The **SRs** of the **SFIEs** (3.1), under the condition (3.2), when the kernel takes a generalized potential function form are:

$$\lambda_2 \sum_{j=0}^{\ell} \int_0^{\ell} \frac{u^{1+m}\, K_m^{v}(u,v)}{(1-u^2)^{\sigma^-}} \cdot {}^{(m)}P_{n_j}^{(m,-\sigma^-)}(1-2u^2)\, du + \lambda_1 \int_0^1 u^{1+m} K_m^{(v)}(u,v) \cdot {}^{(m)}P_{n_\ell}^{(m,-\sigma^-)}(1-2u^2)\, du$$

$$= \lambda_2 \sum_{j=0}^{\ell} u_j\, \mu_{n_j} F_{j,\,k} \cdot {}^{(m)}P_{n_j}^{(m,-\sigma^-)}(1-2u^2) + \lambda_1 u_{n_\ell} \cdot {}^{(m)}P_{n_\ell}^{(m,-\sigma^-)}(1-2u^2), \qquad (3.27)$$

$$\mu_{n_\ell} = 2^{2\sigma^+}\,\Gamma(n_\ell + \sigma^+)\,\Gamma(2n_\ell + \sigma^+)\,[\,n_\ell!\,\Gamma(1+2n_\ell)\,]^{-1}$$

$$\left(\sigma^{\pm} = \frac{1 \pm w}{2},\ v = w + \frac{1}{2},\ 0 \le w < \frac{1}{2}\right).$$

Many special cases can be derived from (3.25) as the following:

(i) Carleman kernel, $m = \pm\frac{1}{2}$

(ii) Logarithmic kernel, $v = \frac{1}{2}$, $m = \pm\frac{1}{2}$,

(iii) Elliptic kernel, $v = \frac{1}{2}$, $m = 0$

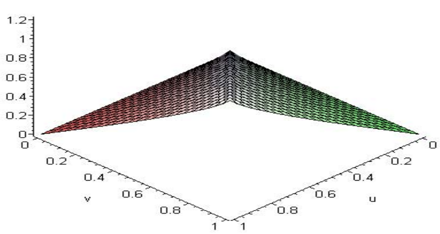

Fig. 1, m=0.5, v=0.1

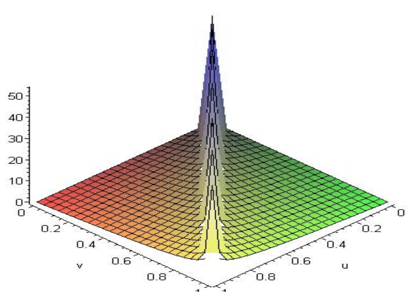

Fig.2, m=0.5, v=0.55

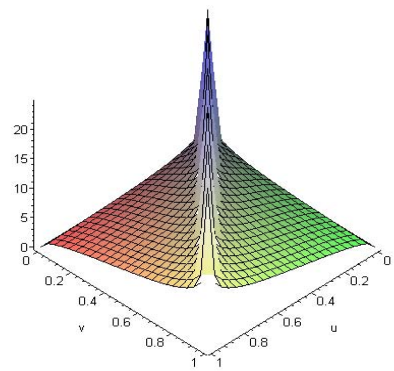

Fig.3

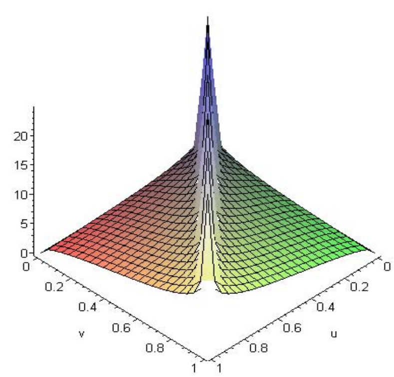

Fig.4

Theorem 5: The **SRs** of **SFIEs** (3.1), under the condition (3.2) for the complete elliptic kernel can be obtained in the form:

$$\lambda_2 \sum_{j=0}^{\ell} u_j\, F_{j,\ell} \int_0^{\ell} \frac{u\, E\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)}{\sqrt{1-\rho^2}} P_{2\,m_j}\left(\sqrt{1-\rho^2}\right) d\rho + \lambda_1 \int_0^1 \frac{u\, E\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)}{\sqrt{1-\rho^2}} P_{2\,m_\ell}\left(\sqrt{1-\rho^2}\right) d\rho$$

$$= \lambda_2 \frac{\pi^2}{4} \sum_{j=0}^{\infty} u_j F_{j,\ell} \left[\frac{\left(2\,m_j - 1\right)\,!!}{\left(2m_j\right)\,!!}\right] P_{2m_j}\left(\sqrt{1-r^2}\right) + \lambda_1 \frac{\pi^2}{4}\left[\frac{\left(2m_\ell - 1\right)\,!!}{\left(2m_\ell\right)\,!!}\right] P_{2m_\ell}\left(\sqrt{1-r^2}\right)$$

($P_m^{(0,-\frac{1}{2})}\left(1-2x^2\right) = P_{2m}\left(\sqrt{1-x^2}\right)$, $P_m(z)$ is a Legendre polynomial) (3.29)

The importance of the integral equation with complete elliptic kernel came from the work of Kovalenko [17], who developed the **FIE** of the first kind for the mechanics mixed problem of continuous media and obtained an approximate solution for the **FIE** of the first kind with complete elliptic kernel.

(iv) Potential kernel, ν=0.5

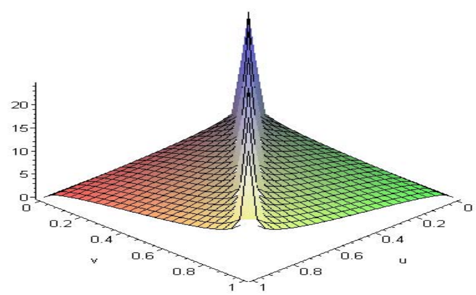

Fig. 5, m=0.1

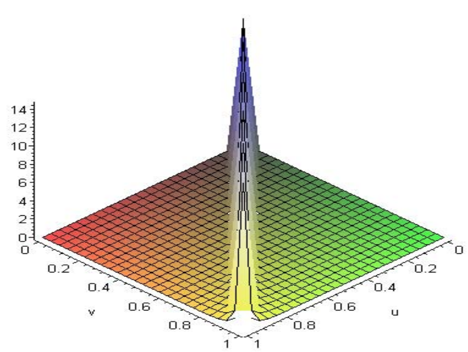

Fig. 6, m=50.

(v) Generalized potential kernel

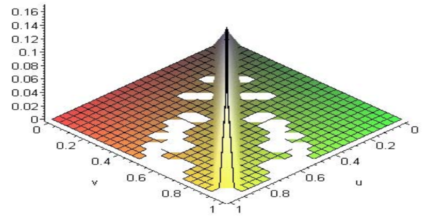

Fig. 7, m=50, ν=0.2

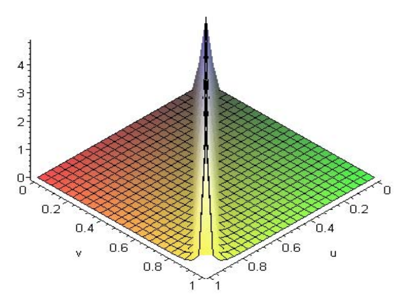

Fig.8 , m=120, ν=o.4

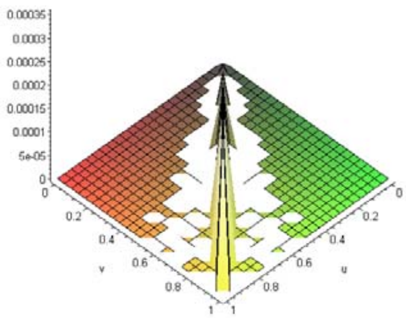

Fig. 9, m=200, ν=0.01

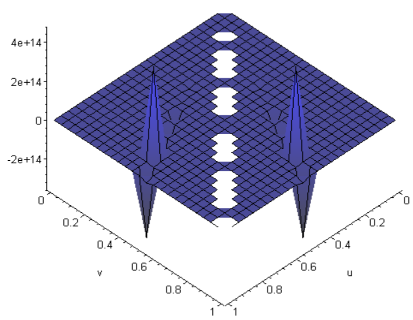

Fig.10 , m=120,ν=o.5

Many important spectral relations can be derived and established from the formula (3.25), for different values of ν, $0 \le \nu < 1$ and for higher order m_j, $j = 0, 1, 2, \ldots, \ell$.

4. Conclusion and results

From the above results and discussion, the following may be concluded

(1) The contact problem of a rigid surface of an elastic material, when a stamp of length $2a$ is impressed into an elastic layer surface of a strip by a variable $P(t), 0 \le t \le T \le 1$, whose eccentricity of application $e(t)$, see [11], becomes special case of this work.

(2) The numerical method used transforms the **MIE** into **SFIEs**.

(3) The **SFIEs** depends on the number of derivatives of $F(t,\tau)$ with respect to time t, $t \in [0,T], T \le 1$.

(4) The displacement problems of anti plane deformation of an infinite rigid strip with width $2a$, putting on an elastic layer of thickness h is considered as a special case of this work when $t = 1$, $F(t,\tau) = 1$, $f(x,t) = H$ and $\varphi(x,1) = \psi(x)$. Here, H represents the displacement magnitude and $\psi(x)$ the unknown function represents the displacement stress, see [18].

(5) The problems of infinite rigid strip with width $2a$ impressed in a viscous liquid layer of thickness h , when the strip has a velocity resulting from the impulsive force $V = V_0 e^{-iwt}, i = \sqrt{-1}$, where V_0 is the constant velocity , w is the angular velocity resulting rotating the strip about z-axis are considered as special case of this work , when $F(t,\tau) =$ constant and $t = 1$, see [18] .

(6) In the above discussion (4) and (5) and when $h \to \infty$, this means that the depth of the liquid (fluid mechanics) or the thickness of elastic material (contact problem) becomes an infinite .

(7) The three kinds of the displacement problem, in the theory of elasticity and mixed contact problems, which discussed in [11,18] ,are considered special cases of this work .

(8) The generalized potential kernel represents a Weber-Sonin integral formula (3.25) and represents a non homogeneous wave equations. The kernel (3.25) can be written in the Legendre polynomial form as follows

$$K_m^{\alpha}(u,v) = 2^{-2w^{-}} (uv)^{m+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma^2(n+m+1-w^-) P_n^m(u) P_n^m(v)}{\Gamma^2(n+m+1)(2n+m+1-w^-)^{-1}}$$

($P_n^m(u)$ is Legendre polynomial and $W^{\pm} = \dfrac{1 \pm \alpha}{2}$).

(9) Taking in mind the basic relations of Bessel function, the generalized potential kernel (3.25) satisfies the following nonhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \rho^2}\right) K_m^{\nu}(r,\rho) = \left(h(r) - h(\rho)\right) K_m^{\nu}(r,\rho),$$

$$h(r) = \left(m^2 - \frac{1}{4}\right) r^{-2}, \quad \left(m \ne \pm \frac{1}{2}\right)$$

(10) This paper is considered as a generalization of the worker of the contact problems in continuous media for the Fredholm integral equation of the first and second kind when the kernel takes the following forms: Logarithmic kernel, Carleman kernel, elliptic integral kernel, and potential kernel. Moreover the contact problem which leads us to the integro-differential equation with Cauchy kernel is contained also as a special case, see [1]. Also in this work the contact problems of higher-order ($m \geq 1$) harmonic are included as special cases, see [1-8, 11-15, 17-19].

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