Three-dimensional flow over a stretching surface in a viscoelastic fluid with mass and heat transfer

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Abstract: This paper presents a study of the laminar three-dimensional flow of non-Newtonian incompressible viscoelastic fluid with mass and heat transfer over an infinite horizontal stretching sheet under heat generation (absorption) and chemical reaction. The governing differential equations which describe the motion of the problem are converted into dimensionless formulas by using a similarity transformation method and solved analytically by using The Kummer's function. The parameters of viscoelastic, internal heat generation /absorption, chemical reaction and the dimensionless stretching ratio are included and discussed numerically in the governing equations of momentum, energy and concentration. The effects of the elasticity, heat, reaction effect and the stretching ratio parameters with Prandtl and Schmidt numbers on the velocity, temperature and concentration distributions have been discussed and illustrated graphically. [Nature and Science 2010;8(8):218-228]. (ISSN: 1545-0740).

Keywords: Three-dimensional flow; viscoelastic fluid; mass; heat transfer

1. Introduction

The study of the flow field due to a stretching surface in an ambient fluid is important in several practical applications in the field of metallurgy and chemical engineering. A number of technical processes concerning polymers involve the cooling of continuous strips or of filaments by drawing them through a quiescent fluid. Extrusion processes, fibers spinning, hot rolling, manufacturing of plastic and rubber sheet, continuous casting and glass blowing are examples of industrial applications of stretching of a surface in an ambient fluid. In these which are governed by the structure of the boundary layer near the moving strip (see [1]), Sakiadis [2] was probably the first to study the two-dimensional boundary layer flow due to a stretching surface in a fluid at rest. Since then, many investigators have studied various aspects of this problem such as heat and mass transfer, constant or variable wall temperature or wall heat flux, the effect of surface suction or blowing, the effect of the magnetic field, etc. for a Newtonian fluid. The case of the three-dimensional boundary layer flow of a Newtonian fluid caused by a stretching flat surface in two lateral directions in an otherwise ambient fluid has been considered by Wang [3],Devietal. [4], Lakshmisha et al. [5], Takhar et al. [6], Aboeldahab and Azzam [7], Chamkha [8] and Edabe et. al. [9].

Ever increasing industrial applications in the manufacture of plastic film and artificial fiber materials, in recent years, has led to a renewed interest in the study of a viscoelastic fluid. Good lists of references on this problem can be found in Sadeghy and Sharifi [10], Khan et al. [11], and Hayat et al. [12] However, these studies the steady laminar boundary layer flow of a viscoelastic fluid due to the stretching of a flat surface in two lateral directions through an otherwise quiescent fluid has been considered and solved by using the homotopy analysis method (HAM). The problem of three-dimensional boundary layer flow of a viscoelastic fluid due to a stretching surface has been considered before as Hayat [13] without mass and heat transfer. also see G. Nath [14] discussed the unsteady three-dimensional stagnation point flow of a viscoelastic fluid. Here the steady laminar boundary layer flow of a viscoelastic fluid with mass and heat transfer in three dimensional stretching sheet, further more, the effects of heat generation/absorption and chemical reaction are discussed.

2. Formulation of the problem

Consider the three-dimensional flow of an incompressible viscoelastic fluid bounded by a stretching surface. Under the usual boundary layer approximations the flow is governed by the following equations:

The continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

The momentum equations

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\upsilon\frac{\partial^2 u}{\partial z^2}-k_o\left(\begin{array}{l} u\dfrac{\partial^3 u}{\partial x\,\partial z^2}+w\dfrac{\partial^3 u}{\partial z^3}- \\ \left(\begin{array}{l}\dfrac{\partial u}{\partial x}\dfrac{\partial^2 u}{\partial z^2}+\dfrac{\partial u}{\partial z}\dfrac{\partial^2 w}{\partial z^2}+ \\ 2\dfrac{\partial u}{\partial z}\dfrac{\partial^2 u}{\partial x\,\partial z}+2\dfrac{\partial w}{\partial z}\dfrac{\partial^2 u}{\partial z^2}\end{array}\right)\end{array}\right) \qquad (2)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\upsilon\frac{\partial^2 v}{\partial z^2}-k_o\left(\begin{array}{l} v\dfrac{\partial^3 v}{\partial y\,\partial z^2}+w\dfrac{\partial^3 v}{\partial z^3}- \\ \left(\begin{array}{l}\dfrac{\partial v}{\partial y}\dfrac{\partial^2 v}{\partial z^2}+\dfrac{\partial v}{\partial z}\dfrac{\partial^2 w}{\partial z^2} \\ +2\dfrac{\partial v}{\partial z}\dfrac{\partial^2 v}{\partial y\,\partial z}+2\dfrac{\partial w}{\partial z}\dfrac{\partial^2 v}{\partial z^2}\end{array}\right)\end{array}\right) \qquad (3)$$

The energy equation

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\kappa\frac{\partial^2 T}{\partial z^2}+Q\left(T-T_\infty\right) \qquad (4)$$

The Concentration equation

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D\frac{\partial^2 C}{\partial z^2}+\kappa_1\left(C-C_\infty\right) \qquad (5)$$

Where u, v *and* w are the velocities in the x-, y- and z-directions, respectively, ρ is the fluid density, $\upsilon=\mu/\rho$ is the Kinematic viscosity, μ is the dynamic viscosity, k_o is the material fluid parameter, c_p is the specific heat at constant pressure, κ is the thermal conductivity, T is temperature of the fluid flow, C is the mass concentration of the species of the flow, Q is the volumetric rate of heat generation/absorption, D is the molecular diffusion coefficient, T_∞ and C_∞ are the fluid temperature and concentration far away and κ_1 is the reaction rate coefficient.

3. Solution of momentum equation

The boundary conditions corresponding to the momentum equations (2) and (3) are given by

$$u=u_w(x)=a\,x,\; v=v_w(y)=b\,y \quad \text{and} \quad w=0 \quad \text{at} \quad z=0 \qquad (6)$$

$$u\to 0,\; v\to 0,\; \frac{\partial u}{\partial z}\to 0 \quad \text{and} \quad \frac{\partial v}{\partial z}\to 0 \quad \text{as} \quad z\to\infty \qquad (7)$$

where a *and* b are positive constants. We write

$u=a\,x\,f\,'(\eta),\; v=b\,y\,g\,'(\eta),$

And $w=-\sqrt{a\,\upsilon}\left(f(\eta)+g(\eta)\right)$ (8)

Where

$$\eta=\sqrt{\frac{a}{\upsilon}}\,z \qquad (9)$$

Here f *and* g are the dimensionless stream functions, η is the similarity variable and prime denotes the differentiation with respect to η.

In order to reduce equations (2) and (3) to dimensionless form by using the transformations in equations (8) and (9) as

$$f'''-f'^2+(f+g)f''+K\left[\begin{array}{l}(f+g)f^{iv} \\ +(f''-g'')f'' \\ -2(f'+g')f'''\end{array}\right]=0, \qquad (10)$$

$$g'''-g'^2+(f+g)g''+K\left[\begin{array}{l}(f+g)g^{iv} \\ +(g''-f'')g'' \\ -2(f'+g')g'''\end{array}\right]=0, \qquad (11)$$

The corresponding boundary conditions (6) and (7) reduce to

$$f(0)=0,\; g(0)=0,\; f'(0)=1 \quad \text{and} \quad g'(0)=c, \qquad (12)$$

$$f'(\infty)=0,\; g'(\infty)=0,\; f''(\infty)=0 \quad \text{and} \quad g''(\infty)=0, \qquad (13)$$

Where $K=k_o a/\upsilon$ is the dimensionless viscoelastic parameter and $c=b/a$ is the dimensionless stretching ratio.

For $c=0$ the problem reduces to the two-dimensional case described by

$$f'''-f'^2+ff''+K\begin{bmatrix} ff^{iv} \\ +f''^2- \\ 2f'f''' \end{bmatrix}=0, \qquad (14)$$

With the boundary conditions

$$f(0)=0, f'(0)=1, f''(\infty)=0 \quad \text{and} \quad f'(\infty)=0, \qquad (15)$$

The analytical solution of equation (14) satisfying equation (15) is

$$f(\eta)=\frac{1-e^{-\delta\eta}}{\delta}, \qquad \delta=\frac{1}{\sqrt{1-K}} \qquad (16)$$

For $0 \le K < 1$

For $c=1$ (axisymmetric flow), we the equations (10) and (11) reduce to

$$f'''-f'^2+2ff'' +2K\left[ff^{iv}-2f'f'''\right]=0, \qquad (17)$$

With the same boundary conditions as in equation (15).

Now let us seek a solution of equations (10) and (11) in the form

$$f(\eta)=\sqrt{\frac{K(1+c)}{c}}\left(1-e^{-\sqrt{1+2c}\,\eta}\right) \qquad (18)$$

$$g(\eta)=\sqrt{\frac{c^2}{1+2c}}\left(1-e^{-\sqrt{\frac{c}{K(1+c)}}\eta}\right) \qquad (19)$$

with the condition

$$2Kc^2+(3K-1)c+K=0 \qquad (20)$$

This is satisfied by the boundary conditions (12) and (13).

On substituting (18) and (19) into (8), the velocity components takes the form

$$u=ax\sqrt{\frac{K(1+c)(1+2c)}{c}}\left(e^{-\sqrt{1+2c}\,\eta}\right) \qquad (21)$$

$$v=ay\sqrt{\frac{c^3}{1+2c}\left(\frac{1}{K(1+c)}\right)}\left(e^{-\sqrt{\frac{c}{K(1+c)}}\eta}\right) \qquad (22)$$

$$w=-\sqrt{a\upsilon}\left(\sqrt{\frac{K(1+c)}{c}}\left(1-e^{-\sqrt{1+2c}\,\eta}\right)+\sqrt{\frac{c^2}{1+2c}}\left(1-e^{-\sqrt{\frac{c}{K(1+c)}}\eta}\right)\right) \qquad (23)$$

The skin-friction coefficients along the x- and y-directions and the heat transfer coefficient can be expressed in the form

$$C_f=\frac{\tau_{wx}}{\rho u_w^2}=\left(\mathrm{Re}_x\right)^{-1/2}f''(0) \qquad (24)$$

$$\tilde{C}_f=\frac{\tau_{wy}}{\rho u_w^2}=\left(\mathrm{Re}_x\right)^{-1/2}\frac{v_w}{u_w}g''(0) \qquad (25)$$

$$N_u=\frac{x\left(\frac{\partial T}{\partial z}\right)_{z=0}}{\left(T_\infty-T_w\right)}=\left(\mathrm{Re}_x\right)^{-1/2}g'(0) \qquad (26)$$

Where

$$\tau_{wx}=\mu\left(\frac{\partial u}{\partial z}\right)_{z=0}, \qquad \tau_{wy}=\mu\left(\frac{\partial v}{\partial z}\right)_{z=0}$$

and $$\left(\mathrm{Re}_x\right)=\frac{u_w x}{\upsilon} \qquad (27)$$

Here C_f and $\tilde{C}_f$ are the skin-friction coefficients along the x- and y-directions, respectively, N_u is Nusselt number represents the heat transfer, τ_{wx} and τ_{wy} are the shear stresses in the x- and y-directions, respectively, μ is the coefficient of viscosity and Re_x is the local Reynolds number.

4. Heat transfer

The governing boundary layer three dimensional equation with temperature-dependent heat generation (absorption) as above equation (4), the thermal boundary conditions, depend on the type of heating process under consideration, are considered by

$$T=T_w=T_\infty+A\left(\frac{x}{l}\right)^2 =T_\infty+B\left(\frac{y}{l}\right)^2 \quad at \quad z=0 \qquad (28)$$

$$T\to T_\infty \quad as \quad z\to\infty \qquad (29)$$

where A and B are two constants and l is the characteristic length. Defining the non-dimensional temperature

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty} \qquad (30)$$

And using the relations (8), (9) and (30), equation (4) and the boundary conditions (28) and (29) can be written as

$$\theta''(\eta)+p_r\left(f(\eta)+g(\eta)\right)\theta'(\eta) -\left(2p_r\left(f'(\eta)+g'(\eta)\right)-\alpha\right)\theta(\eta)=0 \qquad (31)$$

$$\theta(\eta=0)=1, \quad \theta(\eta=\infty)=0 \qquad (32)$$

where $p_r = {}^{\kappa}\!/_{\upsilon \rho c_p}$ is the Prandtl number and $\alpha = {}^{Q}\!/_{a \rho c_p}$ is the heat source (sink) parameter. On substituting (18)-(20) into (31), we can get

$$\theta''(\eta) + p_r\left(\frac{1+c}{m}\right)\left(1-e^{-m\eta}\right)\theta'(\eta) -\left(2p_r\left((1+c)e^{-m\eta}\right)-\alpha\right)\theta(\eta)=0 \quad (33)$$

where

$$m = \frac{c}{K(1+c)} \quad (34)$$

Introducing the change of variable

$$-\frac{(1+c)}{m^2}e^{-m\eta} = \xi \quad (35)$$

And interesting (35) in (33) we obtain

$$\xi\theta''(\xi) + \left[1-\frac{p_r(1+c)}{m^2} - p_r\,\xi\right]\theta'(\xi) + p_r\left[2+\frac{\alpha}{m^2\xi}\right]\theta(\xi)=0 \quad (36)$$

Thus, the solution of equation (33) satisfying (32) is given by

$$\theta(\eta) = e^{-\left(\frac{p_r(1+c)+\lambda}{2m}\right)\eta}\frac{M\left[\varepsilon, \frac{1+\lambda}{m^2}, \chi e^{-m\eta}\right]}{M\left[\varepsilon, \frac{1+\lambda}{m^2}, \chi\right]} \quad (37)$$

where M is the Kummer's function (Abramowitz and Stegun [15]) and it is defined by

$$M(a_o, b_o, z) = 1 + \sum_{n=1}^{\infty}\frac{(a_o)_n\, z^n}{(b_o)_n\, n!}$$

$$(a_o)_n = a_o(a_o+1)(a_o+2),...,(a_o+n-1)$$

$$(b_o)_n = b_o(b_o+1)(b_o+2),...,(b_o+n-1) \quad (38)$$

and

$$\lambda = \sqrt{p_r}\sqrt{(1+c)^2 p_r - 4m^2\alpha},$$

$$\varepsilon = \frac{-4m^2 + p_r(1+c)+\lambda}{2m^2},$$

$$\chi = -\frac{p_r(1+c)}{m^2} \quad (39)$$

5. Mass transfer

The governing boundary layer three dimensional equation with the effect of chemical reaction as equation (5), the boundary conditions are considered by

$$C = C_w = C_\infty + \bar{A}x^2 = C_\infty + \bar{B}y^2 \quad at \quad z = 0 \quad (40)$$

$$C \to C_\infty \quad as \quad z \to \infty \quad (41)$$

where $\bar{A}$ and $\bar{B}$ are constants. Introducing the similarity transformation

$$\varphi(\eta) = \frac{C - C_\infty}{C_w - C_\infty} \quad (42)$$

By using the relations (8), (9) and (42), equation (5) and the boundary conditions (40) and (41) can be written as

$$\varphi''(\eta) + S_C\left(f(\eta)+g(\eta)\right)\varphi'(\eta) -\left(2S_C\left(f'(\eta)+g'(\eta)\right)-\gamma\right)\varphi(\eta)=0 \quad (43)$$

$$\varphi(\eta=0)=1\ , \quad \varphi(\eta=\infty)=0 \quad (44)$$

Where $S_C = {}^{\upsilon}\!/_{D}$ is the Schmidt number and $\gamma = {}^{\kappa_1}\!/_{a}$ is the chemistry reaction parameter. On substituting (18)-(20) into (43), we can get

$$\varphi''(\eta) + S_C\left(\frac{1+c}{m}\right)\left(1-e^{-m\eta}\right)\varphi'(\eta) -\left(2S_C\left((1+c)e^{-m\eta}\right)-\gamma\right)\varphi(\eta)=0 \quad (45)$$

On substituting (35) into (44), we can get

$$\xi\varphi''(\xi) + \left[1-\frac{S_C(1+c)}{m^2} - S_C\,\xi\right]\varphi'(\xi) + S_C\left[2+\frac{\gamma}{m^2\xi}\right]\varphi(\xi)=0 \quad (46)$$

With the boundary conditions

$$\varphi(\xi=\infty)=1\ , \quad \varphi(\xi=0)=0 \quad (47)$$

Thus, the solution of equation (46) satisfying (47) and using the inverse transformation of equation (35) is given by

$$\varphi(\eta) = e^{-\left(\frac{S_C(1+c)+\beta}{2m}\right)\eta}\frac{M\left[\varepsilon_1, 1+\frac{\beta}{m^2}, \varepsilon_2 e^{-m\eta}\right]}{M\left[\varepsilon_1, 1+\frac{\beta}{m^2}, \varepsilon_2\right]} \quad (48)$$

Where

$$\beta = \sqrt{(1+c)^2 S_C^2 - 4m^2\gamma S_C},$$

$$\varepsilon_1 = \frac{\beta - 4m^2 + S_C(1+c)}{2m^2},$$

$$\varepsilon_2 = -\frac{S_C(1+c)}{m^2} \quad (49)$$

6. Results and Conclusion

The three-dimensional flow of non-Newtonian viscoelastic fluid with heat and mass transfer over infinite stretching sheet under heat generation / absorption and chemical reaction is governed by six parameters, namely, K the viscoelastic parameter, α the heat parameter, γ the chemical reaction parameter, c the dimensionless stretching ratio, P_r the Prandtl number and S_c the Schmidt number. An insight into the effects of these parameters of the flow field can be obtained by the study of the temperature and mass concentration distributions. The dimensionless temperature $\theta(\eta)$ and dimensionless mass concentration $\varphi(\eta)$ have been plotted against the dimension η for several sets of the values of the parameters $K, \alpha, c, \gamma, P_r$ *and* S_C.

Fig. (1) Show that the temperature distribution decreases with an increases in the dimensionless stretching ratio c. In Fig. (2), the variation of the temperature $\theta(\eta)$ with K (the viscoelastic parameter) seen and show that the temperature distribution decreases with an increases of K. Figures (3), (4) and (5) Show that the temperature distribution decrease with an increases in the Prandtl number with $c = 1, 10, 100$ respectively. But in figure (6) the temperature distribution $\theta(\eta)$ increases with an increase in the heat parameter $+ve$ α and the inverse is true seeing figure (7).

In the figures (8), (9) and (10) present the dimensionless mass concentration profiles $\varphi(\eta)$ for selected values of the parameters; c the dimensionless stretching ratio, K the viscoelastic parameter and S_C the Schmidt number respectively with fixed the other parameters. It is shown that the dimensionless mass concentration at a given point in the fluid is decrease with increase the parameters c, K and S_C. From figure (11) one sees that the effect of the chemical reaction parameter γ is increase the dimensionless mass concentration $\varphi(\eta)$ of the fluid flow.

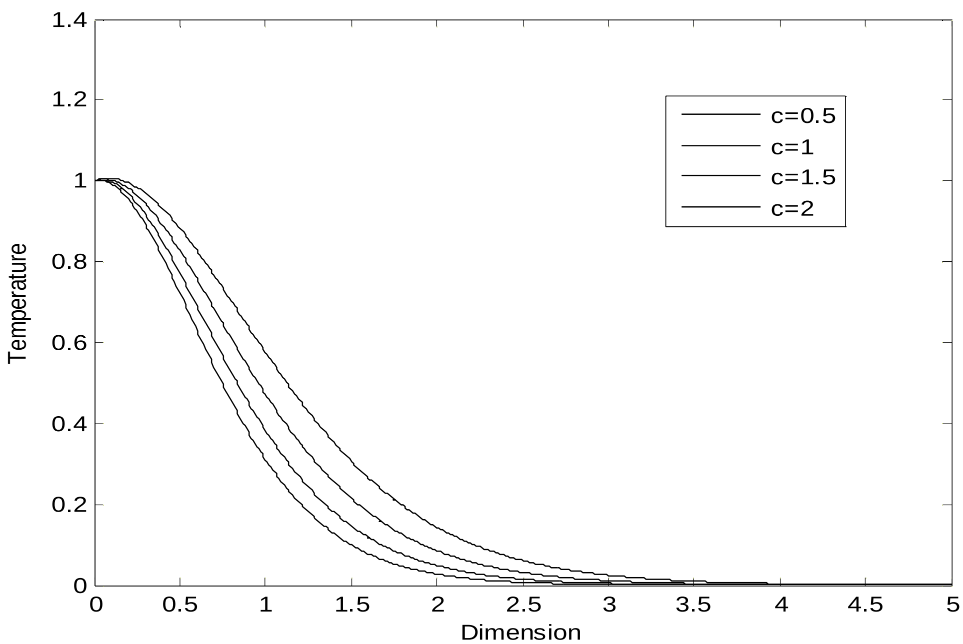

Figure (1): Temperature distribution for varies values of the dimensionless stretching ratio c at $P_r = 1.2$, $\alpha = 0.25$

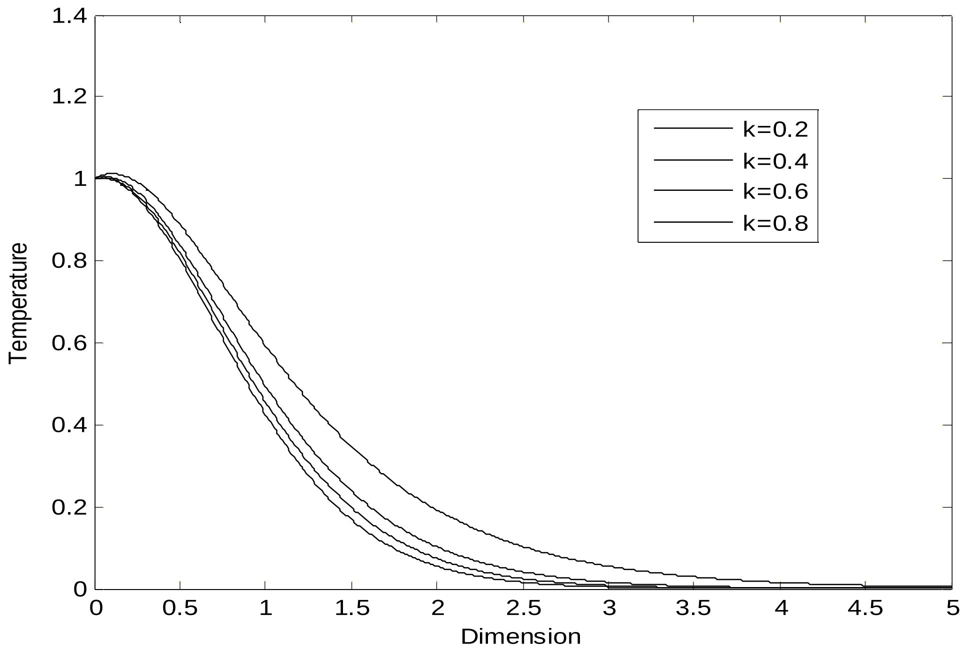

Figure (2): Temperature distribution for varies values of the viscoelastic parameter K at $P_r = 1.2$, $\alpha = 0.25$

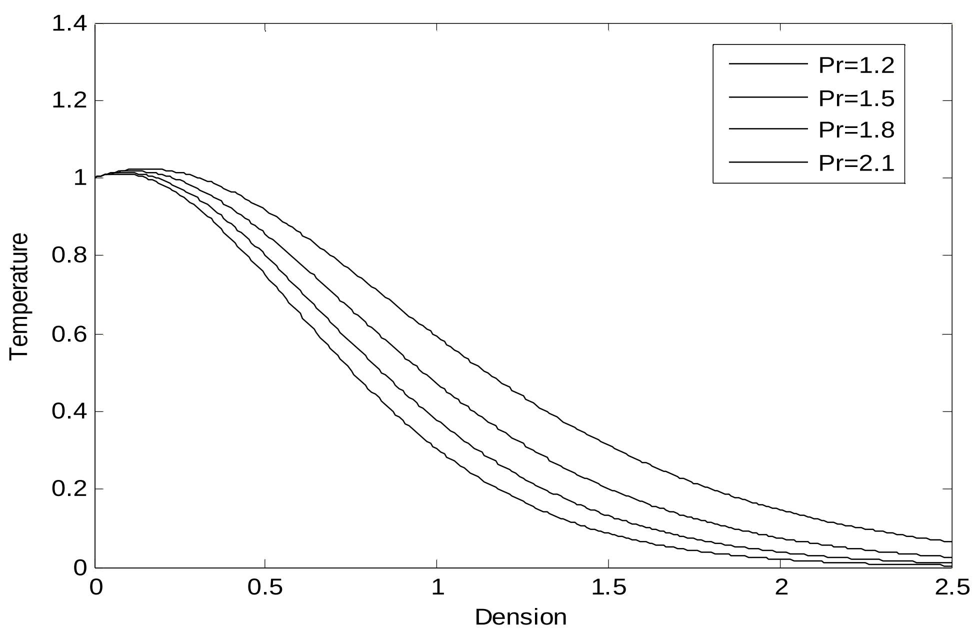

Figure (3): Temperature distribution for varies values of the Prandtl number at $c = 1$, $\alpha = 0.5$

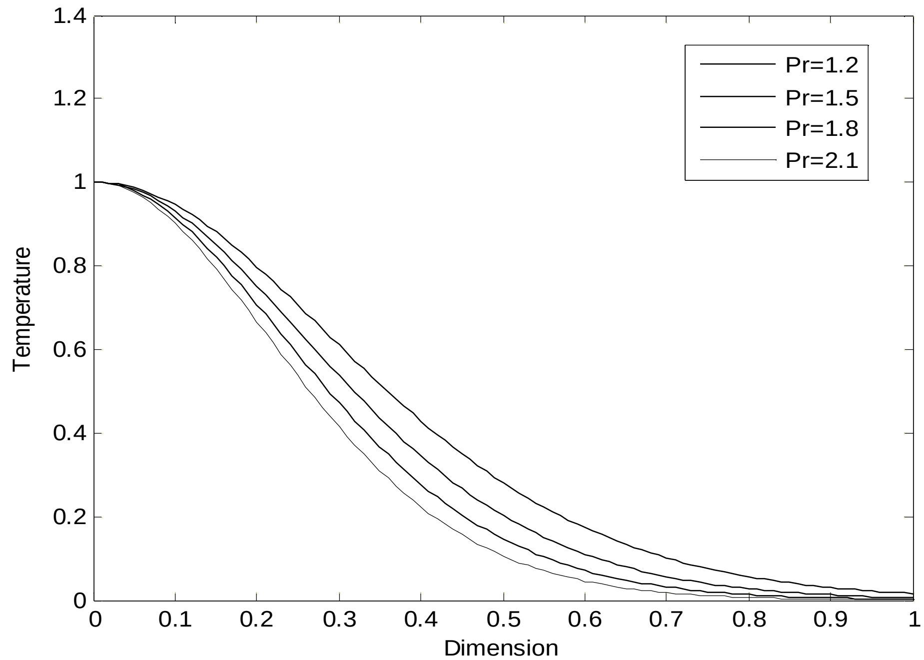

Figure (4): Temperature distribution for varies values of the Prandtl number at $c = 10$, $\alpha = 0.5$

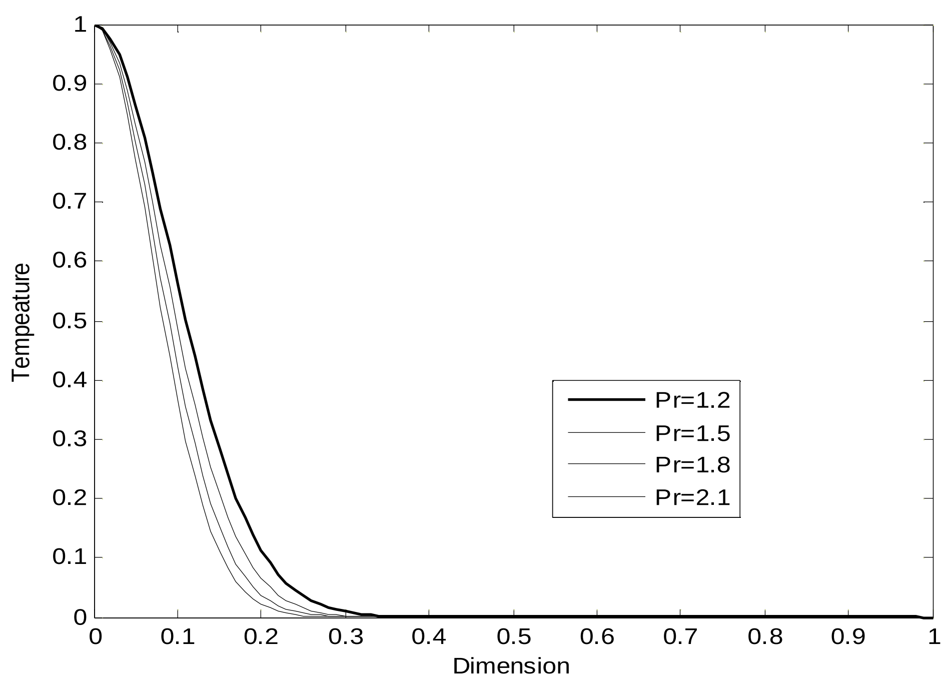

Figure (5): Temperature distribution for varies values of the Prandtl number at $c = 100$, $\alpha = 0.5$

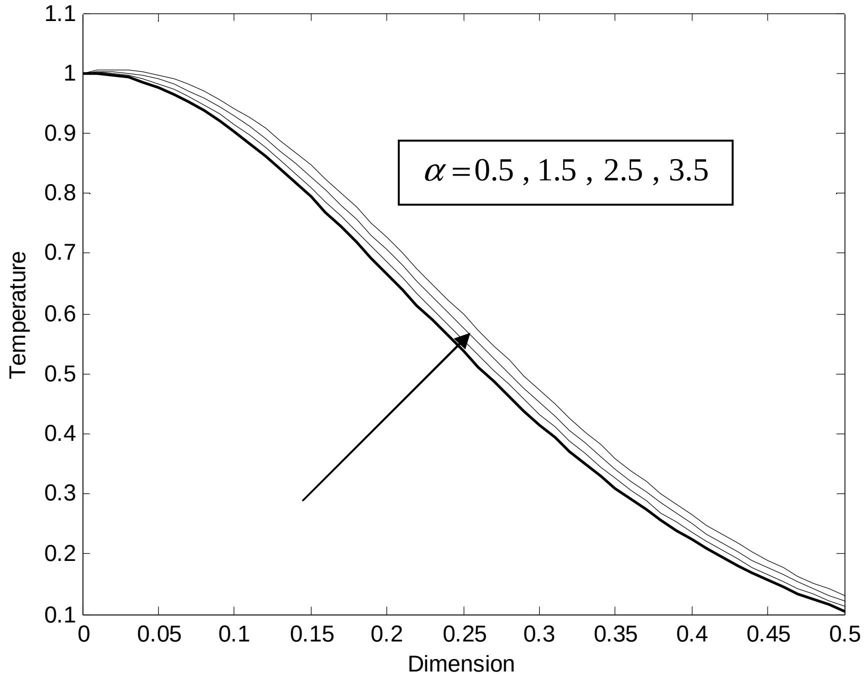

Figure (6): Temperature distribution for varies values of the heat parameter α at $K = 0.4$, $P_r = 2.1$

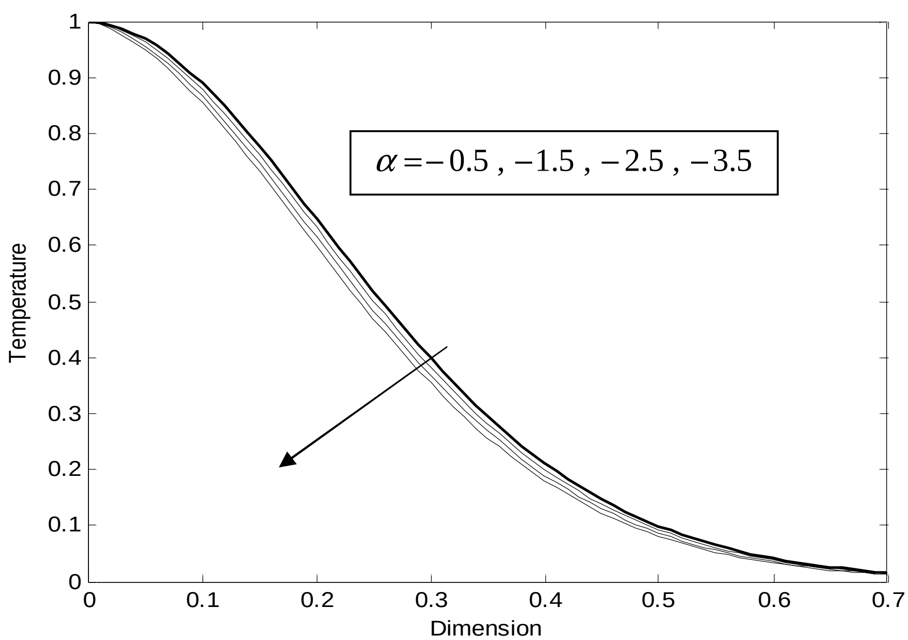

Figure (7): Temperature distribution for varies values of the heat parameter α at $K = 0.4$, $P_r = 2.1$

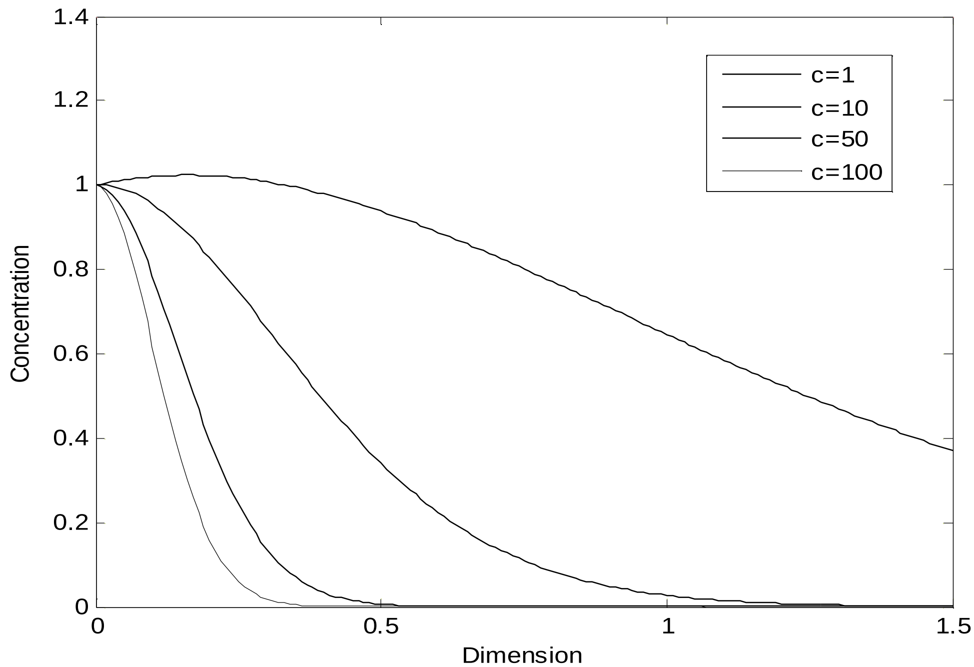

Figure (8): Concentration distribution for varies values of the dimensionless stretching ratio c at $S_C = 1.0, \gamma = 0.5$

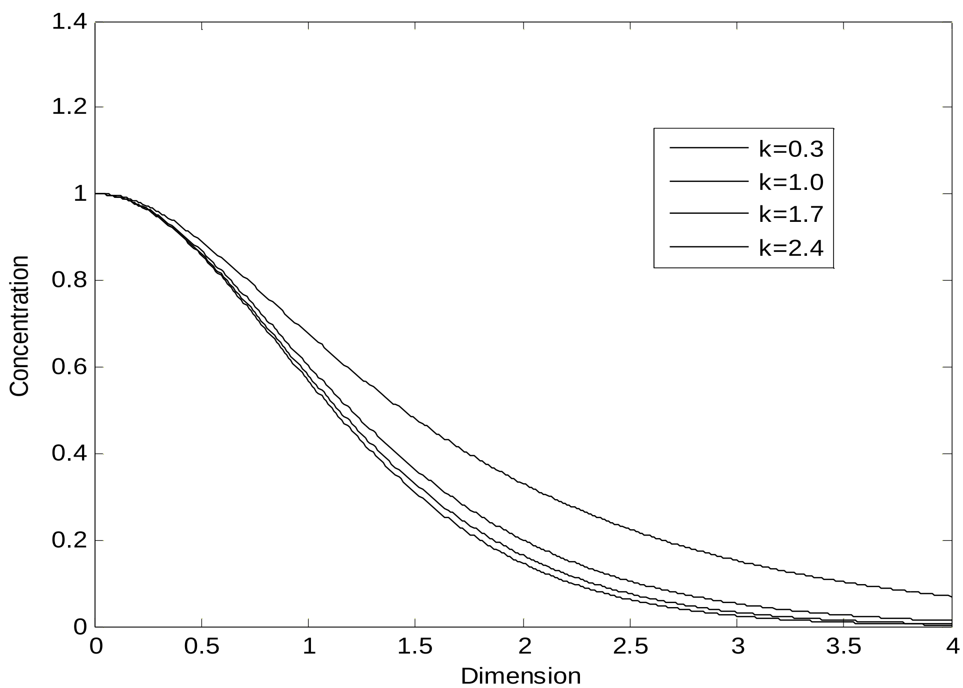

Figure (9): Concentration distribution for varies values of the viscoelastic parameter K at $S_C = 0.2, \gamma = 0.15$

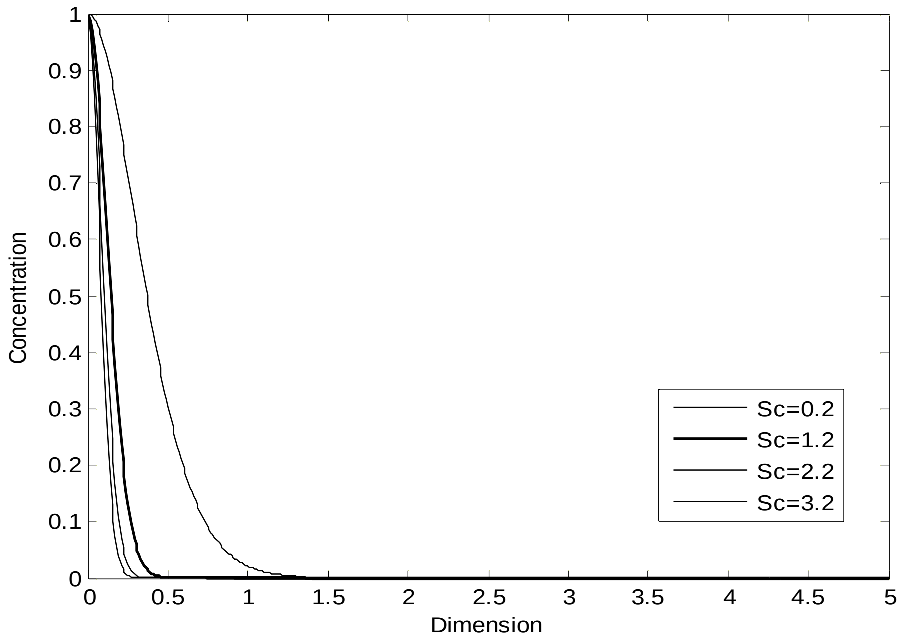

Figure (10): Concentration distribution for varies values of the Schmidt number S_C at $c = 60, \gamma = 0.15$

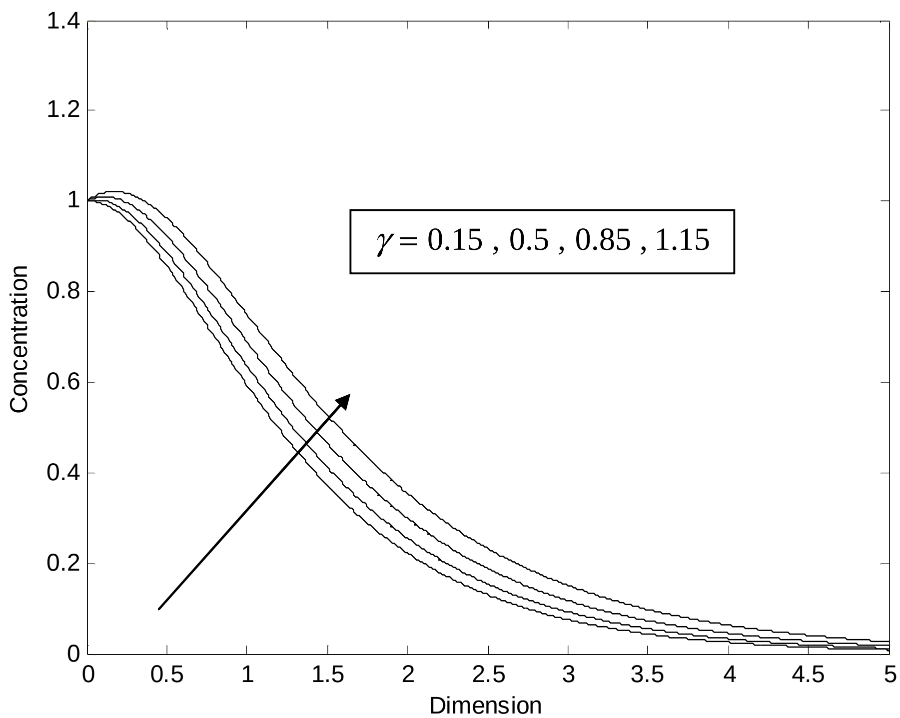

Figure (11): Concentration distribution for varies values of the Chemical reaction parameter γ at $c = 7.5$, $S_C = 0.2$

7. References

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