Properties Standard Frame in Hilbert C*- Modules

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Abstract: The goal of the present paper is a short introduction to a general module frame theory in Hilbert C* - modules over a unital C*- algebra. In this paper firstly we recall some basic propertics of Hilbert space, Hilbert modules and modular standard frames then by using adjointable module homomorphism on Hilbert C*- modules and on $l^2(A)$. we construct some frames. Finally we present a relation between standard frame in Hilbert C*- modules. We also study the behavior of Bessel sequences and frames under operators. In addition, we obtain a relation between standard frames in Hilbert C* - modules. We focus on finitely and countably generated Hilbert *A*-module over unital C* - algebra *A* and Our references are [1] and [6].

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1. Introduction

Hilbert space frames were originally introduced by Duffin and Schaffer[2] to deal with some problems in non-harmonic Fourier analysis. Hilbert C^*- modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*- algebra rather than in the field of complex numbers [4]. These frames are called *Hilbert* C^*- *modular frames* or just simply *modular frames*. These concepts are generalizations of some results in [11].

In this paper firstly we recall some basic propertics in Hilbert C^*- modules, secondly by using adjointable module homomorphism on Hilbert C^*- modules and on $\ell^2(A)$ we construct some frames and finally we present a relation between standara frames in Hilbert C^*- modules. Our references for Hilbert space are [1] and [6].

2.Perliminaries

We review some basics about Hilbert C^*- modules and Hilbert C^*- modular frames.

For basic notaions and theory for Hilbert C^*-modules see[5,7,9,11].

In this paper $\mathbb{N}$ will denote the set of natural numbers and J will be a finite or countable Subset of $\mathbb{N}$.

Definition 2.1. Let *A* be a (unital) C^*-algebra and *H* be a (left)*A*-module. Suppose that The linear structures given on *A* and *H* are compatible, i.e. $\lambda(ax)=a(\lambda x)$ for every $\lambda \in C, a \in A$ and $x \in H$. Assume that there exists a mapping $\langle .,. \rangle : H \times H \to A$ with the properties:

$(i)\langle x,x \rangle \geq 0$ for every $x \in H$,

$(ii)\langle x,x \rangle = 0$ if and only if $x = 0$,

$(iii)\langle x,y \rangle = \langle y,x \rangle^*$ for every $x,y \in H$,

$(iv)\langle ax,y \rangle = a\langle x,y \rangle$ for every $x,y \in H$, and every $a \in A$,

$(v)\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$ for every $x,y,z \in H$.

Then the pair $\{H,\langle .,. \rangle\}$ is called a (left)pre-Hilbert *A*-module. The map $\langle .,. \rangle$ is said to be an *A*-valued inner product. If the pre-Hilbert *A*-module $\{H,\langle .,. \rangle\}$ is complete with respect to the induced norm $\|x\| = \|\langle x,x \rangle\|^{\frac{1}{2}}$ then it is called a Hilbert *A*-module.

In this paper we focus on finitely and countably generated Hilbert A-module over unital C^*-algebra A. In case A is unital the Hilbert A-module H is *(algebraically) finitely generated* if there exists a finite set $\{x_1,...,x_n\}\subseteq H$ such that every element $x\in H$ can be expressed as an A-linear combination $x=\sum_{j=1}^{j=n} a_j x_j,\ a_j\in A$. A Hilbert A-module H is *countably generated* if there exists a countable set $\{x_j\}_{j\in N}\subseteq H$ such that the set of all finite A-linear combinations $\left\{\sum_j a_j x_j\right\}, a_j\in A$, is norm-dense in H.

Definition 2.2.(see [4]) Let A be a unital C^*-algebra and J be a finite or countable index set. A (finite or countable) sequence $\{x_j\}_{j\in J}$ of elements in a Hilbert A-module H is said to be a *frame* for H if there exist two constants $C,D\rangle 0$ such that for every $x\in H$

$$C\langle x,x\rangle\le\sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle\le D\langle x,x\rangle \quad (1)$$

If for every $x\in H$, the series in the middle of the inequality (1) is convergent in norm, we say that the frame is standard. The numbers C and D are called *frame bounds*. Likewise, sequence $\{x_j\}_{j\in J}$ is called a (standard) *Bessel sequence* with Bessel bound D if there exists D>0 such that

$$\sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle\le D\langle x,x\rangle$$

The sequence $\{x_j\}_{j\in J}$ satisfies the lower frame bound if there exists a C>0

$$C\langle x,x\rangle\le\sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle$$

The frame $\{x_j\}_{j\in J}$ is saide to be a *tight frame* if C=D, and said to be *normalized* if C=D=1.

We consider standard (normalized tight) frames on finitely or countably generated Hilbert A-module over unital C^*- algebra A. For a unital C^*-algebra A, let $\ell^2(A)$ be the Hilbert A-module, see[4], define by

$$\ell^2(A)=\left\{\{a_j\}_{j\in J}\subset A:\sum_{j\in J}a_ja_j^*\ converges\ in\ \|.\|\right\}$$

For any standard frame $\{x_j\}_{j\in J}$ of a finitely or countably generated Hilbert A-module H, the frame transform of the frame $\{x_j\}_{j\in J}$ is definded to be the map

$$\theta(x):H\to\ell^2(A)\quad,\quad \theta(x)=\left\{\langle x,x_j\rangle\right\}_{j\in J}$$

that is bounded, A-linear and adjointable with adjoint

$$\theta^*(x):\ell^2(A)\to H\quad,\quad \theta^*(e_j)=x_j$$

for a standard basis $\{e_j\}_{j\in J}$ of the Hilbert A-module $\ell^2(A)$ and all $j\in J$. See[5] Moreover for every $x\in H$,

$$|\theta(x)|^2=\langle\theta(x),\theta(x)\rangle=\sum_{j\in J}\langle x,x_j\rangle\langle x_j,x\rangle$$

Therefore θ is one-to-one with a closed range which is complemented in $\ell^2(A)$, $\ell^2(A)=\theta(H)+Ker\theta^*(x)$. We note that $\theta^*|\theta(H)$ is an invertible operator and the frame operator $S=\left(\theta^*\theta\right)^{-1}$ is a positive invertible bounded operator on H such that for every $x\in H$,

(2)

$$x=\sum_{j\in J}\langle x,Sx_j\rangle x_j=\sum_{j\in J}\langle x,x_j\rangle Sx_j$$

The sequence $\{Sx_j\}_{j\in J}$ is a frame for H and is called the *canonical dua l frame* of $\{x_j\}_{j\in J}$.

Now suppose that $\{x_j\}_{j\in J}$ is a Bessel sequence of a finitely or countably generated Hilbert *A*-module *H,* the associated *analysis operator* $T_X : H \to \ell^2(A)$ is defined by

$$T_X x = \sum_{j\in J} \langle x, x_j \rangle e_j \qquad x \in H.$$

Note that analysis operator T_X is adjointable and adjoint T^*_X fulfills $T^*_X e_j = x_j$ for all *j.*

Throughout this paper, we denote by $\hat{L}(H,K)$, the set of all adjointable maps from *H* to *K* and as usual we abbreviate $\hat{L}(H,H)$ by $\hat{L}(H)$.

3.Construction of frames in Hilbert C^*-module

Lemma 3.1. Let *H* and *K* be Hilbert C^*-modules.

(*i*) If $\{x_j\}_{j\in J}$ is a Bessel sequence in *H* with bound *D* and $T \in \hat{L}(H,K)$, then $\{Tx_j\}_{j\in J}$ is a Bessel sequence in *K* with bound $D\|T\|^2$,

(*ii*) If $\{x_j\}_{j\in J}$ satisfies the lower frame condition and there exists a positive constant *B* such that for every $y \in \overline{T(H)}$, $B\langle y,y\rangle \le \langle T^*y, T^*y\rangle$, then $\{Tx_j\}_{j\in J}$ satisfies the lower frame condition in $\overline{T(H)}$.

Proof. (*i*) By proposition 1.2 of [8] for every *y*∈*K* we have

$$\sum_{j\in J} \left|\langle y, Tx_j\rangle\right|^2 = \sum_{j\in J} \left|\langle T^*y, x_j\rangle\right|^2 \le D\langle T^*y, T^*y\rangle \le |D|\|T\|^2\langle y,y\rangle$$

(*ii*) For every $y \in \overline{T(H)}$ we have

$$CB\langle y,y\rangle \le C\langle T^*y, T^*y\rangle \le \sum_{j\in J} \left|\langle T^*y, x_j\rangle\right|^2 = \sum_{j\in J} \left|\langle y, Tx_j\rangle\right|^2$$

In the following theorem we give a necessary and suffient condition for $\{y_j = Tx_j\}_{j\in J}$ to be standard frame of $\overline{T(H)}$.

Theorem 3.2. Let $\{x_j\}_{j\in J}$ be a standard frame of *H* with bounds 0<C≤D and T be a module map in $\hat{L}(H,K)$, Then the statements are equinalent.

(*i*) The sequence $\{Tx_j\}_{j\in J}$ is a standard frame of $\overline{T(H)}$;

(*ii*) There exists a positive costant *B* such that T^* satisfies:

(3) $B\langle y,y\rangle \le \langle T^*y, T^*y\rangle$

For every $y \in \overline{T(H)}$.

Proof. Suppose that $\{Tx_j\}_{j\in J}$ is a standard frame of $\overline{T(H)}$ with lower bound C'.

Then for every $y \in \overline{T(H)}$,

$$C'\langle y,y\rangle \le \sum_{j\in J} \left|\langle y, Tx_j\rangle\right|^2 = \sum_{j\in J} \left|\langle T^*y, x_j\rangle\right|^2$$

(4) $\le D\langle T^*y, T^*y\rangle$

From which, condition (3) follow with $B = C'/D$.

The converse follows from the above lemma. Moreover since $\{x_j\}_{j\in J}$ is a standard frame

$\sum_{j\in J}\left|\left\langle y,Tx_j\right\rangle\right|^2$ is convergent in norm, so $\sum_{j\in J}\left|\left\langle y,Tx_j\right\rangle\right|^2$ is convergent in norm for every $y\in\overline{T(H)}$.

In previous theorem if $\{Tx_j\}_{j\in J}$ is a standard frame in *K*, then by the reconstruction formula (2), $T(H)$ is dense in *K*, so for $\{Tx_j\}_{j\in J}$ to be a standard frame of *K* it is necessary than $T(H)$ be dense in *K* and consequently the assumption $\overline{T(H)}=K$ yields the following result.

Corollary 3.3. Let $\{x_j\}_{j\in J}$ be a standard frame of *H* with bounds 0<C≤D and T be a module map in $\hat{L}(H,K)$ such that $\overline{T(H)}=K$. Then the statements are equinalent.

(*i*) The sequence $\{Tx_j\}_{j\in J}$ is a standard frame of K;

(*ii*) There exists a positive costant *B* such that T^* satisfies:

(5) $B\langle y,y\rangle\le\langle T^*y,T^*y\rangle$

for every $y\in K$.

Remark. (*a*) There exists a Hilbert *A*- module *H* and a module map $T\in\hat{L}(H)$ such that T^* is injective, but $\overline{T(H)}\neq H$.(*cf.*[10], *Exercise* 15.F). For this reason, in corollary (3.3) we supposed that $\overline{T(H)}=K$. But if $\overline{T(H)}$ is a complemented submodule of *H* then condition (5) implise that $\overline{T(H)}=H$.

(*b*) If *T* is a self adjoint module map in $\hat{L}(H)$ and satisfies condition (5), then *T* is invertible(*cf.* [7], *Lemma* 3.1). In particular *T* is surjective. Then $\overline{T(H)}=H$.

(*c*) Suppose that $T\in\hat{L}(H)$ and $T(H)$ is closed. Then $T(H)$ is a complemented submodule of *H* and $T(H)\oplus KerT^*=H$ (*cf.* [8], *Theorem* 3.2). if T^* satisfies condition (5), then $KerT^*=\{0\}$ and $T(H)=H$.

Corollary 3.4. Let $\{x_j\}_{j\in J}$ be a standard frame of *H*. If *T* is an adjointable module map from *H* onto *K*, then the statements are equivalent.

(*i*) The sequence $\{y_j=Tx_j\}_{j\in J}$ is a standard frame of K;

(*ii*) There exists a positive costant *B* such that T^* satisfies:

6) $B\langle y,y\rangle\le\langle T^*y,T^*y\rangle$

for every $y\in K$.

By Corollary 3.4, we can construct some standard frames for a closed submodule of *H*, with a given standard frame.

Now, let $T\in\hat{L}(\ell^2(A))$ and let $\eta=\{\eta_j\}_{j\in J}$ be a standard frame of *H* with bounds C_η and D_η and frame transform θ_η. We use *T* to construct the sequence $\xi=\{\xi_j\}_{j\in J}$, where

(7) $\xi_j=\theta^*(T(e_j))\quad,\quad(j\in J)$

Such that

$$T(e_j)=\sum_{j\in J}a_{ji}e_j\quad,\quad\{a_{ji}\}_{j\in J}\in\ell^2(A)$$

then

$$\theta_\eta^*(T(e_j))=\sum_{j\in J}a_{ji}\theta_\eta^*(e_i)=\sum_{j\in J}a_{ji}\eta_i$$

But the sequence $\xi=\{\xi_j\}_{j\in J}$ is not always a standard frame for *H* (e.g. T=0).

Now we want to make $\xi=\{\xi_j\}_{j\in J}$ a standard frame under an appropriate condition on *T*.

Theorem 3.5. Let $\{\eta_j\}_{j\in J}$ be a standard frame of *H* with bounds C_η and D_η. If $T\in\hat{L}(\ell^2(A))$ then the following statements are equivalent.

(*i*) The sequence $\{\xi_j\}_{j\in J}$ is a standard frame of H defined by (7) ;

(*ii*) There exists a positive costant B such that T^* satisfies:

$$B\langle y,y\rangle \le \langle T^*y,T^*y\rangle$$

for every $y\in\theta_\eta(H)$ where is the frame transform of $\{\xi_j\}_{j\in J}$.

Proof. Suppose that $\{\xi_j\}_{j\in J}$ is a standard frame for H defined by (7). Then there are constants $0<C_\xi\le D_\xi$ such that for every $x\in H$,

$$C_\xi\langle x,x\rangle \le \sum_{j\in J}\langle x,\xi_j\rangle\langle \xi_j,x\rangle \le \langle \theta_\xi(x),\theta_\xi(x)\rangle$$

(8) $\le D_\xi\langle x,x\rangle$

Where θ_ξ is the frame transform of $\{\xi_j\}_{j\in J}$. Also for every $n\in J$ and $x\in H$,

$$\langle \theta_\xi(x),e_n\rangle=\langle x,\theta_\xi^*(e_n)\rangle=\langle x,\xi_n\rangle=\langle x,\theta_\eta^*(T(e_n))\rangle$$
$$=\langle T^*(\theta_\eta(x)),e_n\rangle$$

Since the set of A-linear combinations of $\{e_j\}_{j\in J}$ is dense in $\ell^2(A)$, we have

(9) $\theta_\xi=T^*\theta\eta$

So, by using the left inequality of (8), and (9), we conclude that

$$C_\xi\langle x,x\rangle\le\langle\theta_\xi(x),\theta_\xi(x)\rangle$$
$$=\langle T^*\theta_\eta(x),T^*\theta_\eta(x)\rangle$$

for every $x\in H$. Then

$$\frac{C_\xi}{D_\eta}\langle\theta_\eta(x),\theta_\eta(x)\rangle\le C_\xi\langle x,x\rangle$$
$$\le\langle T^*\theta_\eta(x),T^*\theta_\eta(x)\rangle$$

for every $x\in H$. Therefore sufficient we take $B=\dfrac{C_\xi}{D\eta}$.

Conversely, by using Proposition 1.2 of [8], since for every $x\in H$, $\sum_{j\in J}|\langle x,\eta_j\rangle|^2$ is convergent in A, we have for every $x\in H$,

$$C_\eta B\langle x,x\rangle\le B\langle\theta_\eta(x),\theta_\eta(x)\rangle\le\langle T^*\theta_\eta(x),T^*\theta_\eta(x)\rangle$$
$$=\sum_{j\in J}|\langle x,\xi_j\rangle|^2\le\|T\|^2\langle\theta_\eta(x),\theta_\eta(x)\rangle=\|T\|^2\sum_{j\in J}|\langle x,\eta_j\rangle|^2$$
$$\le\|T\|^2D_\eta\langle x,x\rangle.$$

There fore $\{\xi_j\}_{j\in J}$ is a frame with frame transform

$$\theta_\xi=T^*\theta_\eta.$$

4. Relation between standard frames in Hilbert *A*-module

The aim of this section is to characterize all standard frames of *H*. In theorem 4.2, we will show how any two standard frames of H are related with each other.

Definition 4.1. Frames $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ of H and K, respectively, are similar if there is an A-linear adjointable, bounded operator $T:H\to K$ such that for each $j\in J$, $T(x_j)=y_j$ and T is invertible.

Theorem 4.2. Let $\{\eta_j\}_{j\in J}$ and $\{\xi_j\}_{j\in J}$ be a standard frames in H and K, respectively, then they are similar. Conversely, if $\{\eta_j\}_{j\in J}$ is a standard frame in H and $\{\xi_j\}_{j\in J}$ is a frame in K which is similar to $\{\eta_j\}_{j\in J}$, then $\{\xi_j\}_{j\in J}$ also is standard.

Proof. If θ_η and θ_ξ are transforms frames for $\{\eta_j\}_{j\in J}$ and $\{\xi_j\}_{j\in J}$, respectively, then $\theta_\eta(H)$ and $\theta_\xi(H)$ are complemented in $\ell^2(A)$. Therefore the orthogonal projections $p:\ell^2(A)\to\theta_\eta(A)$ and $q:\ell^2(A)\to\theta_\xi(A)$ are adjointable. If we take $T=\theta_\xi^{-1}\circ p\circ\theta_\eta:H\to K$, then T is an A-linear, bounded adoptable operator with such that for each $j \in J$, $T^*(\xi_j)=\eta_j$ and similarly the map $U=\theta_\eta^{-1}\circ q\circ\theta_\xi : K\to H$ is adjoitable with $U^*=\theta_\eta^*\circ q\circ\theta_\eta^{*-1} : H\to K$ such that for each $j\in J$, $U^*(\eta_j)=\xi_j$. Hence $U^*T^*=id_K$ and $T^*U^*=id_H$. Therefore we have the result. The converse is obvious.

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