Inequality of Nikolsky and Bernshteins's type classification within $H_p(-\infty,\infty)$

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Abstract: In this work the analytical functions in the upper semi plane are learned. Therefore, there received inequalities for norms of Hardy space $H_p(-\infty,\infty)$, which is the analogue in some ideas inequality of S.M.Nokolsiy and S.N.Bernstein.

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1. Introduction.

Let $H_p = H_p(-\infty,\infty)$ is a space of analytical in the upper semi plane functions $f(z) = f(x+iy),\ \ y>0$ meeting the condition

$$T_p(f;y)=\left\{\int_{-\infty}^{\infty}\left|f(x+iy)^p\,dx\right|\right\}^{\frac{1}{p}}<\infty,\quad 0<p<\infty$$

$$T_\infty(f;y)=\sup_x\left|f(x+iy)\right|<\infty,\quad p=\infty,\qquad -\infty\le x\le\infty.$$

Let $L_p(-\infty,\infty)$ means a space of all measured on $(-\infty,\infty)$ functions for which

$$\|f(x)\|_{L_p}=\left\{\int_{-\infty}^{\infty}|f(x)|^p\,dx\right\}^{\frac{1}{p}}<\infty,\ 0<p<\infty \tag{A}$$

and when $p=\infty$

$$\|f(x)\|_{L_\infty}=\sup_{-\infty<x<\infty}|f(x)|<\infty.$$

Clearly, if $p\ge 1$, the set L_p is space with the norm defined by (A). If $0<p<1,$ the formula (A) does not define a norm since the triangle inequality is not satisfied. However, in this case L_p is a linear metric space.

For the entire functions of the degree $\le\sigma$ within the space $L_p(-\infty;\infty)$ an inequality (see [1], p.150)

$$\left\|Q_\sigma(x)\right\|_{L_p}\le C\sigma^{\frac{1}{q}-\frac{1}{p}}\left\|Q_\sigma(x)\right\|_{L_p}\quad 1\le p\le q\le\infty \tag{1}$$

is known as Nikolsky's inequality, (see [1], p.137-138) also an inequality

$$\left\|Q_\sigma^{(k)}(x)\right\|_{L_p}<M\sigma^k\left\|Q_\sigma(x)\right\|_{L_p},\quad k=1,2,3,...;\quad 1\le p\le\infty,\quad M-const \tag{2}$$

is known as Bernshtein's inequality.

Let's underline, that an analog of an inequality (2) when $0 < p < 1$ is calculated by the author [2] for natural numbers $k = 1, 2, 3, \ldots$, while for the fractional number $k > 0$ if $0 < p < 1$ in [3]. Some properties of the function $f(x) \in L_p(-\infty, \infty)$ possessing derivatives of a fraction order were investigated by us in works [4] and [5].

2. Problem formulation.

The aim of the work is to receive an analog of an inequality (1), (2) and an analog of one inequality of Hardy-Littlewoods [6] within the spaces $H_p(-\infty, \infty)$.

3. Subsidiary facts.

For proving the basic result the following is urgent.

1. It is known (see [7], formulae (2.7)) that for the function $f(z) \in H_1(-\infty; \infty)$ there is a representation

$$f(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t+iy_0) \frac{y-y_0}{(x-t)^2 + (y-y_0)^2} dt, \qquad y > y_0 \geq 0 . \tag{3}$$

2. The analytical function $f(z) \in H_1(-\infty; \infty)$ in the upper semi plane has a representation (see [7])

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t+iy_1)}{t+iy_1 - z} dt \qquad y>y_1>0 \tag{4}$$

3. Integral

$$\int_{+\infty}^{\infty} \left| f(x+iy) \right|^p dx = \varphi(y) \tag{5}$$

as a function from y does not increase (see [8])

4. The inequality (see [7])

$$\|f'(z)\|_{Hp} \leq \|f(x)\|_{Lp}, \quad y > 0, \quad z = x + iy \tag{6}$$

occurs.

5. If

$$\left\{ \int_0^{2\pi} \left| f(re^{i\varphi}) \right|^p d\varphi \right\}^{\frac{1}{p}} = 0\left\{(1-r)^{-\beta}\right\}, \qquad \beta \geq 0,$$

$$\left\{ \int_0^{2\pi} \left| f(re^{i\varphi}) \right|^q d\varphi \right\}^{\frac{1}{q}} = 0\left\{(1-r)^{\frac{1}{q}-\frac{1}{p}-\beta}\right\}, \quad 0 < p < q, \qquad \beta \geq 0, \tag{7}$$

then correlations received by Hardy and Littlewoods occur (see [9]).

4. Basic results.

Theorem 1. If $f(x+iy) \in H_p(-\infty; \infty), \qquad 0 < p \leq \infty$, then an inequality occurs:

$$T_1(f;y) \leq C(p)(y-y_0)^{1-\frac{1}{p}} T_p(f;y_0), \quad 0<p<1, \quad C(p) = \left(\frac{1}{\pi}\right)^{\frac{1}{p}} \frac{4}{2-p}, \quad y>y_0\geq 0, \tag{8}$$

$$T_q(f;y) \le C(p,q)(y-y_0)^{\frac{1}{q}-\frac{1}{p}}\ T_p(f;y_0),\quad 0<p<q, C(p)=\left(\frac{1}{\pi}\right)^{\frac{1}{p}}\left(\frac{4q}{2q-p}\right)^{\frac{1}{q}}. \tag{9}$$

Let's mark that the constant $0<C(p)<2,\quad 0<p<q<\infty$ and if $q=\infty$, then $C(p,q)=(\pi)^{\frac{1}{p}}$.

Theorem 2. If $f(x+iy)\in H_p(-\infty;\infty),\quad 0<p<\infty$ and there is a derivative of the order k, then an inequality:

$$T_p\left(f^{(k)};y\right) \le C(p,k)(y-y_0)^{-k}\,T_p(f;y_0),\qquad y>y_0>0\quad k=1,2,3,\ldots \tag{10}$$

occurs, when $p\ge 1$ constant $C(p,k)$ doesn't depend on p.

Further $C(p,k)$ means a constant, depending on $p,k.$

Theorem 3. If function $f(z)\in H_p,\quad f'(z)\in H_p(-\infty;\infty)$, then when $y>y_0>0$ an inequality

$$\left\{\int_{-\infty}^{\infty}\left|f'(z)\right|^p dx\right\}^{\frac{1}{p}} \le C\frac{\omega(y-y_0)_{L_p}}{y-y_0},\quad z=x+iy,\quad 1\le p\le\infty \tag{11}$$

where $\omega(\delta;f)_{Lp}$ -is a module of continuity (see [1], p. 174-180) of the boundary function $f(x)$ in $L_p(-\infty;\infty)$, i.e.

$$\omega(\delta;f)_{L_p} = \sup_{u\le\delta}\left\|f(x+u)-f(x)\right\|_{L_p}.$$

From theorem 2 and 3 the following stems:

Corollary fact 1. If the condition

$$T_p(f:y_0)=0(y_0{}^{-\alpha}),\qquad \alpha>0,$$

is fulfilled, then

$$T_p(f^{(k)};y)=0(y_0{}^{-k-\alpha}),\ \ k=1,2,\ldots.;\quad y>2,\ \ y_0>0.$$

This is an analog of one result of Hardy and Littlewoods [10], calculated for periodical functions in the class $H_p(-\pi,\pi)$.

Corollary fact 2. If the boundary function $f(x)\in L_p(-\infty,\infty)$ meets the condition

$$\omega(t;f)_{lp}=0(t^{\alpha}),\qquad 0<\alpha<1,$$

then

$$1)\ \left\|f'(z)\right\|_{H_p}=O\left(y_1^{\alpha-1}\right),\ 1\le p\le\infty,\ y>y_1>0;$$

$$2)\ \left\|f'(z)\right\|_{H_p}=O\left(y_1^{\alpha-1}\right),\ \frac{1}{2}<p<1,\ \alpha<1-\frac{1}{p},\ y>y_0.$$

Inequality (8) and (9) are at Nikolsky's type classification (see (1)). Inequality (10) is of Bernstein's type classification (see (2)).

Let's note that inequality (11) is an analog of inequality calculated by Yu. A. Brudniy and Hopengauz for analytical functions of the unit disk at $p\ge 1$ and at $0<p<\infty$ by E. A. Storojenko and Ya. Valashek [11] (for poly harmonically functions in disk M. F. Timan [12]).

Inverse inequality to inequality (11) for integral functions of the degree ≤σ within the space $L_p(-\infty,\infty)$ gives us lemma 1 in [13].

Proving theorem 1.

In equality (3) we shall replace function $f(x+iy)$ in to functions $[f(x+iy)]^p \in H(-\infty,\infty)$, (see [14], p.101).

$$[f(x+iy)]^p = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[f\left(t+iy_0\right)\right]^p \frac{\left(y-y_0\right)dt}{\left(x-t\right)^2+\left(y-y_0\right)^2}. \tag{12}$$

We shall

$$F_p(t,x,y,y_0) = \frac{1}{\pi}[f(t+iy_0)]^p \frac{(y-y_0)}{(x-t)^2+(y-y_0)^2} \tag{13}$$

from (12)

$$f(x+iy) = [\int_{-\infty}^{\infty} F_p(t,x,y,y_0)dt]^{\frac{1}{p}} \tag{14}$$

stems. Hence it follows

$$\left(\int_{-\infty}^{\infty}\left|f(x+iy)\right|dx\right)^p \le \left\{\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} F_p(t,x,y,y_0)dt\right|^{\frac{1}{p}} dx\right\}^p \tag{15}$$

as $0<p<1$ and $\frac{1}{p}>1$, then applying Minkovsky's inequality in the right part of inequality (15) and considering (13) we receive:

$$\left(\int_{-\infty}^{\infty}\left|f(x+iy_0)\right|dx\right)^p \le \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|F_p(t,x,y,y_0)\right|^{\frac{1}{p}} dx\right]^p dt =$$

$$= \frac{y-y_0}{\pi}\int_{-\infty}^{\infty}\left|f(t+iy_0)\right|^p\left(\int_{-\infty}^{\infty}[(x-t)^2+(y-y_0)^2]^{-\frac{1}{p}} dx\right)^p dt \tag{16}$$

Now, let's estimate integral

$$I = \int_{-\infty}^{\infty}[(x-t)^2+(y-y_0)^2]^{-\frac{1}{p}} dx = \int_{-\infty}^{\infty}[u^2+(y-y_0)^2]^{-\frac{1}{p}} du =$$

$$= 2\int_0^{\infty}[u^2+(y-y_0)^2]^{-\frac{1}{p}} du = 2B, \tag{17}$$

$$B = \int_0^{\infty} = \int_0^{y-y_0} + \int_{y-y_0}^{\infty} = B_1 + B_2,$$

$$B_1 = \int_0^{y-y_0}[u^2+(y-y_0)^2]^{-\frac{1}{p}} du \le (y-y_0)^{-\frac{2}{p}}(y-y_0) = (y-y_0)^{1-\frac{2}{p}};$$

$$B_2 = \int_{y-y_0}^{0} [u^2 + (y-y_0)^2]^{-\frac{1}{p}} du \le \frac{p}{2-p}(y-y_0)^{1-\frac{2}{p}}.$$

Thus, integral (17) is estimated as

$$I \le \frac{4}{2-p}(y-y_0)^{1-\frac{2}{p}} \tag{18}$$

Considering estimates (8) under inequality (16) we find:

$$\left(\int_{-\infty}^{\infty} \left|f\left(x+iy\right)\right| dx\right)^p \le \frac{1}{\pi}\left(\frac{4}{2-p}\right)^p \left(y-y_0\right)^{p-1} \int_{-\infty}^{\infty} \left|f\left(t+iy\right)\right|^p dt,$$

i.e. the theorem is proved for $0<p<1,\ q=1.$ Now, let's proved a general case $0<p<q<\infty$. From equality (14) we have:

$$\left\{\int_{-\infty}^{\infty} \left|f(x+iy)\right|^q dx\right\}^{\frac{1}{q}} = \left\{\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} F_p(t,x,y,y_0)dt\right|^{\frac{q}{p}} dx\right\}^{\frac{p}{q}},$$

as $p>q,\ \dfrac{q}{p}>1$, then, applying generalized inequality of Minkovsky we receive:

$$\left\|f\left(x+iy\right)\right\|_{Hq} \le \left(\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|F_p\left(t,x,y,y_0\right)\right|^{\frac{q}{p}} dx\right]^{\frac{p}{q}} dt\right)^{\frac{1}{p}}.$$

Considering designation (13), from the last inequality we receive

$$\left\|f(x+iy)\right\|_{H_q} \le \left(\frac{y-y_0}{\pi}\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty}\left|f(t+iy)\right|^p \left[\int_{-\infty}^{\infty}\left|(x-t)^2+(y-y_0)^2\right|^{-\frac{q}{p}} dx\right]^{\frac{p}{q}} dt\right)^{\frac{1}{p}} \tag{19}$$

Inner integral in the right part of inequality (19) is estimated in the same way as (17), then, calculating in detail we receive:

$$\int_{-\infty}^{\infty}\left|\left(x-t\right)^2+\left(y-y_0\right)^2\right|^{-\frac{q}{p}} dx \le \frac{4q}{2q-p}\left(y-y_0\right)^{1-\frac{2q}{p}} \tag{20}$$

Considering estimation (20) from inequality (19) we get

$$\left\|f(x+iy)\right\|_{H_q} \le C(p,q)(y-y_0)^{\frac{1}{q}-\frac{1}{p}}\left\|f(t+iy_0)\right\|_{Hp},\quad C(p,q) = \left(\frac{1}{\pi}\right)^{\frac{1}{p}}\left(\frac{4q}{2q-p}\right)^{\frac{1}{q}},\quad q>p>0$$

The theorem is proved for $0<p<q<\infty$.

Let's consider when $q=\infty$. From (12) we got

$$\sup_{-\infty\le x<\infty}\left|f(x+iy)\right| \le \sup_{-\infty\le x\le\infty}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\left|f\left(t+iy_0\right)\right|^p \frac{\left(y-y_0\right)dt}{(x-t)^2+\left(y-y_0\right)^2}\right]^{\frac{1}{p}}.$$

Hence it follows

$$\|f(x+iy)\|_{H_\infty} \le \left(\frac{1}{\pi}\right)^{\frac{1}{p}}(y-y_0)^{-\frac{1}{p}}\|f(t+iy_0)\|_{Hp},$$

i.e. the affirmation of the theorem when q=∞. The theorem is proved completely.

Proving theorem 2.

Applying inequality (4) to functions $(z-iy_0)^{-\lambda} f(z+s)$, where $\lambda>0$, s is a free substantial and $y_0>0$

$$f(z+s)\cdot(z-iy_0)^{-\lambda} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t+s+iy_1)\frac{(t+iy_1-iy_0)^{-\lambda}}{(t+iy_1-z)}dt.$$

Hence, differentiating by z, we find out that

$$f'(z+s)=\lambda(z-iy_0)^{-1}f(z+s)+\frac{1}{2\pi i}(z-iy_0)\int_{-\infty}^{\infty} f(t+iy_1+s)\frac{(t+iy_1-iy_0)^{-\lambda}}{(t+iy_1-z)^2}dt. \tag{21}$$

Supposing $z=iy$ at $0<p<1$ from the last inequality we receive

$$|f'(s+iy)|^p \le \lambda^p (y-y_0)^{-p}|f(s+iy)|^p +$$

$$+\left(\frac{1}{2\pi i}\right)^p (y-y_0)^{\lambda p}\left[\int_{-\infty}^{\infty}|f(t+s+iy_1)|\frac{|t+i(y_1-y)|^{-\lambda}}{|t+i(y_1-y)^2|}dt\right]. \tag{22}$$

Consider integral in the right part (22)

$$J^p = \left[\int_{-\infty}^{\infty}|f(t+s+iy_1)|\frac{|t+i(y_1-y)|^{-\lambda}}{|t+i(y_1-y)^2|}dt\right]^p. \tag{23}$$

Applying theorem 1 when $0<p<1$ for integral (23) we receive:

$$J^p \le C(P)(y_1-y_2)^{p-1}\int_{-\infty}^{\infty}|f(u+iy_2)|^p\left|(u-s)^2+(y_2-y_0)^2\right|^{-\frac{\lambda p}{2}}\left|(u-s)^2+(y_2-y)^2\right|^{-p}du,\ y_1>y_2>0. \tag{24}$$

We choose a substantial $\lambda>0$ so that $\lambda p=2$ and considering that

$$\left|(u-s)^2+(y_2-y)^2\right|^{-p} \le (y-y_2)^{-2p}$$

from inequality (24) we receive

$$J^p \le C(p)(y_1-y_2)^{p-1}(y-y_2)^{-2p}\int_{-\infty}^{\infty}|f(u+iy_2)|^p\frac{du}{(u-s)^2+(y_2-y_0)^2}. \tag{25}$$

Now, from inequalities (22), (23), (25) we find out that

$$\int_{-\infty}^{\infty}|f'(s+iy)|^p ds \le \left(\frac{2}{p}\right)^p (y-y_0)^{-p}\int_{-\infty}^{\infty}|f(s+iy)|^p ds +$$

$$+\left(\frac{1}{2p}\right)(y-y_0)^2 C(p)(y-y_2)^{-2p}\int_{-\infty}^{\infty}|f(u+iy_2)|^p\left(\int_{-\infty}^{\infty}\frac{ds}{(u-s)^2+(y_2-y_0)^2}\right)du. \tag{26}$$

Let's note that

$$\int_{-\infty}^{\infty}\frac{ds}{(u-s)^2+(y_2-y_0)^2}=\pi(y_2-y_0)^{-1} \quad (27)$$

as $y>y_1>y_2>0$ and $y_0>0$ is unconditioned, then supposing that $y_1-y_2=y_2-y_0=\frac{y-y_0}{3}$, where $y_2>y_0>0$, from inequality (26) considering (27), we receive:

$$\int_{-\infty}^{\infty}\left|f(s+iy)\right|^p ds\le\left(\frac{2}{p}\right)^p(y-y_0)^{-p}\int_{-\infty}^{\infty}\left|f(s+iy)\right|^p ds+$$

$$+\left(\frac{1}{2p}\right)^p C(P)(y-y_0)^2\left(\frac{y-y_0}{3}\right)^{p-1}\left(\frac{2(y-y_0)}{3}\right)^{-2p}\left(\frac{y-y_0}{3}\right)^{-1}\int_{+\infty}^{\infty}\left|f(u+iy_2)\right|^p du \quad (28)$$

As integral (5) doesn't increase, then from inequality (28) we receive:

$$\int_{-\infty}^{\infty}\left|f'(s+iy_2)\right|^p ds\le\left[\left(\frac{2}{p}\right)^p+c(p)\left(\frac{1}{2\pi}\right)\left(\frac{1}{3}\right)^{p-2}\left(\frac{2}{3}\right)^{-2p}\right](y-y_0)^{-p}\int_{-\infty}^{\infty}\left|f(u+iy_0)\right|^p du=$$

$$=M(p)(y-y_0)^{-p}\int_{-\infty}^{\infty}\left|f(u+iy_0)\right|^p du,$$

i.e. the theorem is proved for $k=1,\quad 0<p<1.$ Repeating reasoning given above k times; we get the affirmation of the theorem for any k when $0<p<1$. When $1\le p<\infty$ we adduce reasoning similar to $0<p<1$, but in this case when integrating inequality (22) we apply Minkovsky's generalized inequality.

Proving theorem 3.

Let's denote that $\xi=y-y_0$, then from inequality (3) we receive:

$$f'(z)=\frac{\partial f}{\partial x}=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\xi f(x+iy_1)2(x-t)}{\left[(x-t)^2+\xi^2\right]^2}dt=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{(x-t)\xi}{\left[(x-t)^2+\xi^2\right]^2}\left[f(x+iy_0)-f(t+iy_0)\right]dt.$$

Here is considered that

$$\int_{-\infty}^{\infty}\frac{\xi dt}{(x-t)+\xi^2}=\pi,\quad \int_{-\infty}^{\infty}\frac{\xi(x-t)dt}{(x-t)^2+\xi^2}=0.$$

Further, replacing variables $x-t=-u$, we receive:

$$f'(z)=\frac{2\xi}{\pi}\int_{-\infty}^{\infty}\frac{u\left[f(x+u+iy_0)-f(x+iy_0)\right]}{\left(u^2+\xi^2\right)^2}du \quad (29)$$

Let $p\ge1$. Applying Minkovsky's generalized inequality we calculate that

$$\left\|f'(z)\right\|_{Hp}\le\frac{2\xi}{\pi}\int_{-\infty}^{\infty}\frac{|u|\left\|f(x+u+iy_0)-f(x+iy_{.0})\right\|_{H_p}}{\left(u^2+\xi^2\right)^2}du. \quad (30)$$

Stemming from (6) we receive

$$\left\|f'(x+u+iy_0)-f(x+iy_0)\right\|_{H_p}\le\left\|f(x+u)-f(x)\right\|_{L_p}\le\omega(|u|:f)_{L_p}.$$

Under (30) we receive:

$$\left\|f'(z)\right\|_{H_p} \le \frac{2\xi}{\pi}\int_{-\infty}^{\infty}\frac{|u|\omega(|u|;f)_{L_p}}{\left(u^2+\xi^2\right)^2}du = \frac{2\xi}{\pi}\left(\int_0^{\infty}+\int_{-\infty}^{0}\right) = \frac{2\xi}{\pi}(J_1+J_2) \quad (31)$$

Let's consider J_1 and J_2

$$J_1 = \int_0^{\infty}\frac{|u|\omega(|u|:f)_{L_p}}{\left(u^2+\xi^2\right)^2}du = \int_0^{\xi}+\int_{\xi}^{\infty} = A_1 + A_2.$$

Under monotony of the module of continuity we receive:

$$A_1 = \int_0^{\xi}\frac{u\omega(u;f)_{L_p}}{\left(u^2+y^2\right)^2}du \le \frac{\omega(\xi;f)_{L_p}}{\xi^2}.$$

Under continuity module we receive that:

$$A_2 = \int_{\xi}^{\infty}\frac{u\omega(u;f)_{L_p}}{\left(u^2+\xi^2\right)^2}du = \frac{c\omega(y;f)_{L_p}}{\xi^2}.$$

Consequently,

$$J_1 \le A_1 + A_2 \le C\frac{\omega(\xi;f)_{L_p}}{\xi^2}, \quad (32)$$

similarly

$$J_2 \le C\frac{\omega(\xi;f)_{L_p}}{\xi^2} \quad (33)$$

and at last we have, that (see (31), (32) and (33))

$$\left\|f'(z)\right\|_{H_p} \le C\frac{\omega(y-y_0)_{L_p}}{(y-y_0)}, \quad p \ge 1.$$

The theorem is proved.

5. Conclusion.

Let's note that theorem 1, shows the correlation between quantities $T_p(f;y)$ at various parameters of p and q, being an analog of Nikolsky's (1) inequality. Theorem 2 shows the connection between functions $f(x+iy)$ and its derivative $f^{(k)}(x+iy)$, k=1,2,3,... within the spaces $H_p(-\infty,\infty)$, being an analog of Bernstein's (2) inequality. Theorem 3 is an analog of Brudny and Hopengauz's results (consequences 1 and 2 are analogs of Hardy-Littlewoods) received for analytical functions in the unit disk.

Consider, that some issues of approximation functions in spaces $H_p(-\infty,\infty)$, by whole functions of final levels learned in this work [13].

6. Comment.

Theorem 1 and 2 are proved by G.Gaimnazarov and theorem 3 is proved by O.G.Gaimnazarov.

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