On Fuzzy Fundamental Groups and Fuzzy Foldings of Fuzzy Minkowski Space

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Abstract: This paper aims to give a combinatorial characterization of and to construct representations of fuzzy fundamental groups of fuzzy sub manifolds on fuzzy Minkowski space $\tilde{M}^4$ by using some geometrical transformations. The fuzzy fundamental groups of the limit fuzzy folding on $\tilde{M}^4$ are presented. The fuzzy fundamental groups of some types of fuzzy geodesics in $\tilde{M}^4$ are obtained.
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1. Introduction

There are many diverse applications of a certain phenomena for which it is impossible to get a relevant data .It may not be possible to measure essential parameters of a process such as the temperature inside molten glass or the homogeneity of a mixture inside some tanks. The required measurement scale may not exist at all, such as in the case of evaluation of offensive smells, evaluating the taste of foods or medical diagnoses by touching [1-9]. The aim of the present paper is to describe the above phenomena geometrically, specifically concerning with the study of the new types of fuzzy retractions, fuzzy folding and fuzzy deformation retract of fuzzy fundamental groups in $\tilde{M}^4$.

A fuzzy manifold is a manifold which has a physical character. This character is represented by the density function μ, where $\mu \in [0,1]$ [9].

A fuzzy subset $(\underset{\sim}{A}, \mu)$ of a fuzzy manifold $(\underset{\sim}{M}, \mu)$ is called a fuzzy retraction of $(\underset{\sim}{M}, \mu)$ if there exist a continuous map $\tilde{r} : (\underset{\sim}{M}, \mu) \longrightarrow (\underset{\sim}{A}, \mu)$ such as $\tilde{r}(a, \mu(a)) = (a, \mu(a)), \ \forall\, a \in \underset{\sim}{A},\ \mu \in [0,1]$ [7, 8, 10].

A fuzzy subset $(\overline{\underset{\sim}{M}}, \tilde{\mu})$ of a fuzzy manifold $(\underset{\sim}{M}, \mu)$ is called a fuzzy deformation retract if there exists a fuzzy retraction $\tilde{r} : (\underset{\sim}{M}, \mu) \longrightarrow (\overline{\underset{\sim}{M}}, \tilde{\mu})$ and a fuzzy homotopy $\tilde{\varphi} : (\underset{\sim}{M}, \mu) \times I \rightarrow (\underset{\sim}{M}, \mu)$ [11] such as

$$\left.\begin{aligned} \tilde{\varphi}((x,\mu),0) &= (x,\mu) \\ \tilde{\varphi}((x,\mu),1) &= \tilde{r}(x,\mu) \end{aligned}\right\} \quad x \in \underset{\sim}{M}$$

$\tilde{\varphi}((a,\mu),t) = (a,\mu),\ \forall (a,\mu) \in \overline{\underset{\sim}{M}},\ t \in I,\ \mu \in [0,1]$ .Where $\tilde{r}(x, \mu)$ is the retraction mentioned above.

A map $\tilde{\Im} : \tilde{M}^4 \longrightarrow \tilde{M}^4$ is said to be an isometric folding of fuzzy submanifolds in $\tilde{M}^4$ into itself iff for any piecewise fuzzy geodesic path $\gamma : J \rightarrow \tilde{M}^4$ is the induced path $\tilde{\Im} \circ \gamma : J \rightarrow \tilde{M}^4$ of a piecewise fuzzy geodesic and is of the same length as γ, where $J = [0,1]$. If $\tilde{\Im}$ does not preserve lengths and $\tilde{\Im}$ is a topological folding of fuzzy submanifolds in $\tilde{M}^4$ [13-19].

2. Main results

Theorem 1. The fuzzy fundamental group of types of fuzzy deformation retracts of $\tilde{M}^4$ is either isomorphic to $\tilde{Z}$ or is a fuzzy identity group.

Proof. Now we will prove that $\tilde{S}_1^1$ and $\tilde{S}_2^1$ are the fuzzy deformation retracts of open fuzzy Minkowski space $\tilde{M}^4$. Consider the fuzzy Buchdahi space $\tilde{B}^4$ [13, 16, 19, 20]. With used fuzzy cylindrical coordinates $z_{(\eta)}, r_{(\eta)}, \theta_{(\eta)}$ and $t_{(\eta)}$ with fuzzy metric

$$ds^2 = -\gamma^2(\gamma^{-1}dr^2(\eta) + r^2(\eta)d\theta^2(\eta) + r^2(\eta)\sin^2\theta(\eta)d\phi^2(\eta)) + p^{-1}dt^2(\eta) \quad (1)$$

Since $\gamma = \frac{1}{2}\ln BC$, if B=C then $\gamma = \ln BC$,

where $\ln B = 1$ implies $\gamma = 1$. Hence (1) becomes

$$ds^2 = -dr^2(\eta) - r^2(\eta)d\theta^2(\eta) - r^2(\eta)\sin^2\theta(\eta)d\phi^2(\eta) + p^{-1}dt^2(\eta) \quad (2)$$

Which is the fuzzy metric of $\tilde{M}^4$. The fuzzy cylindrical coordinates of $\tilde{M}^4$ is given by

$$\tilde{x}_1 = \frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})$$
$$\tilde{x}_2 = \frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})$$
$$\tilde{x}_3 = \frac{s_1}{1-i}\cos(\frac{s_2}{1-ir(\eta)}) \quad (3)$$
$$\tilde{x}_4 = \frac{s_4}{1-\sqrt{p-1}}$$

Where s_1, s_2, s_3 and s_4 are the constant of integration. Solving the Lagrangian equations we obtain fuzzy geodesics and retractions in $\tilde{M}^4$ given by

a fuzzy hypersphere $\tilde{S}_1^1$, $-\tilde{x}_1^2 - \tilde{x}_2^2 - \tilde{x}_3^2 + \tilde{x}_4^2 = 0$, on the null cone and fuzzy great circle $\tilde{S}_2^1$,

$$-\tilde{x}_1^2 - \tilde{x}_2^2 - \tilde{x}_3^2 + \tilde{x}_4^2 = (\frac{s_1}{1-i})^2 + (\frac{s_4}{1-\sqrt{p-1}})^2$$

in $\tilde{M}^4$ and also fuzzy great sphere $\tilde{S}_1^2$,

$$-\tilde{x}_1^2 - \tilde{x}_2^2 - \tilde{x}_3^2 + \tilde{x}_4^2 = (\frac{s_1}{1-i})^2 + (\frac{s_4}{1-\sqrt{p-1}})^2$$

in $\tilde{M}^4$.

The fuzzy deformation retract of $\tilde{M}^4$ is given by

$\tilde{\xi} : (\tilde{M}^4 - \{\tilde{\mu}_i\}) \times I \to (\tilde{M}^4 - \{\tilde{\mu}_i\})$,

where $(\tilde{M}^4 - \{\tilde{\mu}_i\})$ is the open fuzzy Minkowski space $\tilde{M}^4$ and I is the closed interval [0, 1].

The fuzzy retraction is of $(\tilde{M}^4 - \{\tilde{\mu}_i\})$ is $\tilde{R} : (\tilde{M}^4 - \{\tilde{\mu}_i\}) \to \tilde{S}_1^1, \tilde{S}_2^1, \tilde{S}_1^2$.

The fuzzy deformation retract of $(\tilde{M}^4 - \{\tilde{\mu}_i\})$ onto a fuzzy retraction $\tilde{S}_1^1 \subset \tilde{M}^4$ is given by

$$\tilde{\xi}(m,c) = \cos\frac{\pi c}{2}\{(\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}), \frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}), \frac{s_1}{1-i}\cos(\frac{s_2}{1-ir(\eta)}), \frac{s_4}{1-\sqrt{p-1}}\} + \sin\frac{\pi c}{2}\{0,0,0,0\}$$

where

$$\tilde{\xi}(m,0) = \{(\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}), \frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}), \frac{s_1}{1-i}\cos(\frac{s_2}{1-ir(\eta)}), \frac{s_4}{1-\sqrt{p-1}}\}$$, and

$$\tilde{\xi}(m,1) = \{0,0,0,0\}.$$

The fuzzy deformation retract of $(\tilde{M}^4 - \{\tilde{\mu}_i\})$ onto a fuzzy retraction $\tilde{S}_2^1 \subset \tilde{M}^4$ is defined as

$$\tilde{\xi}(m,c)=\frac{1-c}{1+c}\{(\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})$$

$$\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})$$

$$\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),\frac{s_1}{1-i}\cos(\frac{s_2}{1-ir(\eta)}),$$

$$\frac{s_4}{1-\sqrt{p-1}}\}+c(2c-1)\{0,0,(\frac{s_1}{1-i}),(\frac{s_4}{1-\sqrt{p-1}})\}$$

The fuzzy deformation retract of $(\tilde{M}^4-\{\tilde{\mu}_i\})$ onto a fuzzy retraction $\tilde{S}_1^2\subset\tilde{M}^4$ is defined as

$$\tilde{\xi}(m,c)=\ln e^{(1-c)}\{(\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})$$

$$\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})$$

$$\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),\frac{s_1}{1-i}\cos(\frac{s_2}{1-ir(\eta)}),$$

$$\frac{s_4}{1-\sqrt{p-1}}\}+\ln e^{c}\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0$$

$$,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),(\frac{s_4}{1-\sqrt{p-1}})\}$$

Thus, $\pi_1(\tilde{M}^4-\{\tilde{\mu}_i\})\cong\pi_1(\tilde{S}_1^1)$ .Therefore $\pi_1(\tilde{S}_1^1)$ is isomorphic to the fuzzy identity group .Also, $\pi_1(\tilde{M}^4-\{\tilde{\mu}_i\})\cong\pi_1(\tilde{S}_2^1)$ is isomorphic to $\tilde{Z}$. Moreover, $\pi_1(\tilde{M}^4-\{\tilde{\mu}_i\})\cong\pi_1(\tilde{S}_1^2)$ is isomorphic to fuzzy identity group.

Corollary 1. The fuzzy fundamental group of the fuzzy deformation retracts of $(\tilde{M}^4-\{\tilde{\mu}_i\})$ induce two chains of fuzzy fundamental groups up and down $(\tilde{M}^4-\{\tilde{\mu}_i\})$.

Theorem 2. The fuzzy fundamental group of the fuzzy folding of $\tilde{S}_2^1\subset\tilde{M}^4$ is either isomorphic to $\tilde{Z}$ or the fuzzy identity group.

Proof. Now, we are going to discuss the fuzzy folding $\tilde{\Im}$ of $\tilde{S}_2^1\subset\tilde{M}^4$

.Let $\tilde{\Im}:\tilde{S}_2^1\subset\tilde{M}^4\to\tilde{S}_2^1\subset\tilde{M}^4$

,where $\tilde{\Im}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)=(|\tilde{x}_1|,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)$.

An isometric fuzzy folding $\tilde{\Im}$ of $\tilde{S}_2^1\subset\tilde{M}^4$ into itself may be defined by

$$\tilde{\Im}\{0,0,\frac{s_1}{1-i},\frac{s_4}{1-\sqrt{p-1}}\}=\{0,0,\frac{s_1}{1-i},\frac{s_4}{1-\sqrt{p-1}}\}$$

This type of fuzzy folding and any fuzzy folding homeomorphic to this type of fuzzy folding does not induce singularity of $\tilde{S}_2^1\subset\tilde{M}^4$ .Then $\pi_1(\tilde{S}_2^1\subset\tilde{M}^4)\cong\tilde{Z}$.

Now, if the fuzzy folding is defined as $\tilde{\Im}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,|\tilde{x}_4|)$.

An isometric fuzzy folding $\tilde{\Im}$ of $\tilde{S}_2^1\subset\tilde{M}^4$ into itself may be defined by

$$\tilde{\Im}\{0,0,\frac{s_1}{1-i},\frac{s_4}{1-\sqrt{p-1}}\}=\{0,0,\frac{s_1}{1-i},\left|\frac{s_4}{1-\sqrt{p-1}}\right|\}$$

This type of fuzzy folding and any fuzzy folding homeomorphic to this fuzzy folding induce singularity of $\tilde{S}_2^1\subset\tilde{M}^4$, and thus $\pi_1(\tilde{S}_2^1\subset\tilde{M}^4)$ is isomorphic to the fuzzy identity group which is two chains of the fuzzy pointing up and down the density function η .

Corollary 2. The fuzzy fundamental group of the fuzzy folding of $\tilde{S}_2^1\subset\tilde{M}^4$ induces two chains of fuzzy fundamental groups up and down $\tilde{S}_2^1\subset\tilde{M}^4$.

Theorem 3. The fuzzy fundamental group of the limit of the fuzzy foldings of $\tilde{S}_1^2$ in $\tilde{M}^4$ is isomorphic to $\tilde{Z}$.

Proof. Now consider the fuzzy great sphere $\tilde{S}_1^2$ of dimension two,

$$-\tilde{x}_1^2-\tilde{x}_2^2-\tilde{x}_3^2+\tilde{x}_4^2=(\frac{s_1}{1-i})^2+(\frac{s_4}{1-\sqrt{p-1}})^2$$

which is a geodesic in $\tilde{M}^4$ and let

$\tilde{\Im}_1 : \tilde{S}_1^2 \subset \tilde{M}^4 \to \tilde{S}_1^2 \subset \tilde{M}^4$ be a fuzzy folding, now we can define a series of fuzzy folding maps by $\tilde{\Im}_2 : \tilde{\Im}_1(\tilde{S}_1^2 \subset \tilde{M}^4 \to \tilde{S}_1^2 \subset \tilde{M}^4)$, $\tilde{\Im}_3 : \tilde{\Im}_2(\tilde{\Im}_1(\tilde{S}_1^2 \subset \tilde{M}^4 \to \tilde{S}_1^2 \subset \tilde{M}^4))$,…, $\tilde{\Im}_n : \tilde{\Im}_{n-1}(\tilde{\Im}_{n-2}(\tilde{\Im}_{n-3}\ldots\tilde{\Im}_2(\tilde{\Im}_1(\tilde{S}_1^2 \subset \tilde{M}^4))\ldots)) \to \tilde{\Im}_{n-1}(\tilde{\Im}_{n-2}(\tilde{\Im}_{n-3}\ldots\tilde{\Im}_2(\tilde{\Im}_1(\tilde{S}_1^2 \subset \tilde{M}^4))\ldots))$, $\lim_{n\to\infty} \tilde{\Im}_n(\tilde{\Im}_{n-1}(\tilde{\Im}_{n-2}(\tilde{\Im}_{n-3}\ldots\tilde{\Im}_2(\tilde{\Im}_1(\tilde{S}_1^2 \subset \tilde{M}^4))\ldots))$ is a fuzzy circle $(\tilde{S}_2^1 \subset \tilde{M}^4)$ of dimension one. Therefore $\pi_1(\tilde{S}_2^1 \subset \tilde{M}^4)$ is isomorphic to $\tilde{Z}$.

Theorem 4. Under the fuzzy folding $\prod_m(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4) = (\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\frac{|\tilde{x}_4|}{m})$ the fuzzy fundamental group of the limit of the fuzzy folding of the fuzzy geodesic $\tilde{S}_1^2 \subset \tilde{M}^4$ is isomorphic to $\tilde{Z}$.

Proof. Now consider the fuzzy geodesic $\tilde{S}_1^2 \subset \tilde{M}^4$ of dimension two and let $\prod_m : \tilde{S}_1^2 \subset \tilde{M}^4 \to \tilde{S}_1^2 \subset \tilde{M}^4$ be given by

$$\prod_m(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4) = (\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\frac{|\tilde{x}_4|}{m}).$$

Then, the isometric chain fuzzy folding of the fuzzy geodesic $\tilde{S}_1^2 \subset \tilde{M}^4$ into itself may be defined by

$$\prod_1 :\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),(\frac{s_4}{1-\sqrt{p-1}})\}$$

$$\to\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\left|\frac{s_4}{1-\sqrt{p-1}}\right|\},$$

$$\prod_2 :\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\left|\frac{s_4}{1-\sqrt{p-1}}\right|\}$$

$$\to\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\left|\frac{\frac{s_4}{1-\sqrt{p-1}}}{2}\right|\}$$

,… ,

$$\prod_m :\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\left|\frac{\frac{s_4}{1-\sqrt{p-1}}}{m-1}\right|\}$$

$$\to\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\left|\frac{\frac{s_4}{1-\sqrt{p-1}}}{m}\right|\}$$

Then we get

$$\lim_{m\to\infty}\prod_m = \{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}),0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),0\}.$$

Thus, $-\tilde{x}_1^2 - \tilde{x}_2^2 - \tilde{x}_3^2 + \tilde{x}_4^2 = (\frac{s_1}{1-i})^2$, is the fuzzy great circle $\tilde{S}_2^1 \subset \tilde{S}_1^2 \subset \tilde{M}^4$ in $\tilde{M}^4$ with $\tilde{x}_2 = \tilde{x}_4 = 0$. Therefore $\pi_1(\tilde{S}_2^1 \subset \tilde{M}^4)$ is isomorphic to $\tilde{Z}$. Also, the fuzzy fundamental group of types of the limit of fuzzy foldings and any fuzzy manifold homeomorphic to the fuzzy great circle $\tilde{S}_2^1 \subset \tilde{S}_1^2 \subset \tilde{M}^4$ is isomorphic to $\tilde{Z}$.

Theorem 5. Under the fuzzy folding $\prod_m(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4) = (\frac{|\tilde{x}_1|}{m},\frac{|\tilde{x}_2|}{m},\tilde{x}_3,\tilde{x}_4)$, the fuzzy fundamental group of the limit of the fuzzy foldings of $\tilde{M}^4$ is isomorphic to fuzzy identity group.

Proof. Now let $\prod_m : \tilde{M}^4 \to \tilde{M}^4$ be given by

$$\prod_m(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4) = (\frac{|\tilde{x}_1|}{m},\frac{|\tilde{x}_2|}{m},\tilde{x}_3,\tilde{x}_4).$$

Then, the isometric chain fuzzy folding of $\tilde{M}^4$ into itself may be defined by

$$\prod_1:\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),$$

$$\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),$$

$$(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}\to$$

$$\{\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,$$

$$\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,\ ,$$

$$(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}$$

$$\prod_2:\{\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,$$

$$\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,$$

$$(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}\to$$

$$\{\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{2}\right|,$$

$$\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{2}\right|,\ ,\ldots,$$

$$(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}$$

$$\prod_m:\{\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m-1}\right|$$

$$\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m-1}\right|,$$

$$(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}\to$$

$$\{\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m}\right|,$$

$$\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m}\right|,$$

$$(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}.$$

Then we get

$$\lim_{m\to\infty}\prod_m=\{0,0,(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}$$

.

Thus,

$$-\tilde{x}_1^2-\tilde{x}_2^2-\tilde{x}_3^2+\tilde{x}_4^2=((\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}))^2$$

$$+(\frac{s_4}{1-\sqrt{p-1}})^2$$

,which is the fuzzy hypersurface $\tilde{M}_2^1$ in $\tilde{M}^4$ with $\tilde{x}_1=\tilde{x}_2=0$ .Therefore $\pi_1(\tilde{M}_2^1\subset\tilde{M}^4)$ is isomorphic to fuzzy identity group and any fuzzy manifold homeomorphic to $\tilde{M}_2^1$.

Theorem 6. Under the fuzzy folding $\prod_m(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)=(\frac{|\tilde{x}_1|}{m},\frac{|\tilde{x}_2|}{m},\frac{|\tilde{x}_3|}{m},\frac{|\tilde{x}_4|}{m})$, the fuzzy fundamental group of the limit of the fuzzy foldings of $\tilde{M}^4$ is the fuzzy identity group .

Proof. Consider the map $\prod_m:\tilde{M}^4\to\tilde{M}^4$ be given by

$$\prod_m (\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4)=(\frac{|\tilde{x}_1|}{m},\frac{|\tilde{x}_2|}{m},\frac{|\tilde{x}_3|}{m},\frac{|\tilde{x}_4|}{m}).$$

Then, the isometric chain fuzzy folding of $\tilde{M}^4$ into itself may be defined by

$$\prod_1:\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),$$

$$\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}),$$

$$(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}),\frac{s_4}{1-\sqrt{p-1}}\}\to$$

$$\prod_2:\{\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,$$

$$\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,$$

$$\left|(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)})\right|,\left|\frac{s_4}{1-\sqrt{p-1}}\right|\}\to$$

$$\{\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,$$

$$\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})\right|,$$

$$\left|(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)})\right|,\left|\frac{s_4}{1-\sqrt{p-1}}\right|\},$$

$$\{\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{2}\right|,$$

$$\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{2}\right|,$$

$$\left|\frac{(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)})}{2}\right|,\left|\frac{\frac{s_4}{1-\sqrt{p-1}}}{2}\right|\}, \ldots,$$

$$\prod_m:\{\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m-1}\right|,$$

$$\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m-1}\right|,$$

$$\left|\frac{(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)})}{m-1}\right|,\left|\frac{\frac{s_4}{1-\sqrt{p-1}}}{m-1}\right|\}\to$$

$$\{\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m}\right|,$$

$$\left|\frac{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)})}{m}\right|,$$

$$\left|\frac{(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)})}{m}\right|,\left|\frac{\frac{s_4}{1-\sqrt{p-1}}}{m}\right|\}.$$

Then we get $\lim_{m\to\infty}\prod_m=\{0,0,0,0\}$. Thus, it is a fuzzy point and the fuzzy fundamental group of a fuzzy point is the fuzzy identity group.

Theorem 7. The fuzzy fundamental group of the end limits of fuzzy foldings of the n-dimensional fuzzy manifold $\tilde{F}^n$ homeomorphic to $\tilde{M}^n$ into itself is the fuzzy identity group.

Proof. Let $\tilde{\xi}_1:\tilde{F}^n\to\tilde{F}^n$ be a type of fuzzy foldings of n-dimensional fuzzy manifold $\tilde{F}^n$. Then, we have the following chains

$$\tilde{F}^n \xrightarrow{\tilde{\xi}_1^1} \tilde{F}_1^n \xrightarrow{\tilde{\xi}_2^1} \tilde{F}_2^n \to \ldots \tilde{F}_{n-1}^n \xrightarrow{\lim_{i\to\infty}\tilde{\xi}_i^1} \tilde{F}^{n-1}$$

$$\tilde{F}^{n-1} \xrightarrow{\tilde{\xi}_1^2} \tilde{F}_1^{n-1} \xrightarrow{\tilde{\xi}_2^2} \tilde{F}_2^{n-1} \rightarrow \ldots \tilde{F}_{n-1}^{\;n-1} \xrightarrow[i\to\infty]{lim\ \tilde{\xi}_i^2} \tilde{F}^{n-2}, \ldots,$$

$$\tilde{F}^{1} \xrightarrow{\tilde{\xi}_1^n} \tilde{F}_1^{1} \xrightarrow{\tilde{\xi}_2^n} \tilde{F}_2^{1} \rightarrow \ldots \tilde{F}_{n-1}^{\;1} \xrightarrow[i\to\infty]{lim\ \tilde{\xi}_i^n} \tilde{F}^{0},$$

The end of the limits of fuzzy folding coincides with the zero-dimensional fuzzy manifold .Thus, it is a fuzzy point and the fuzzy fundamental group of a fuzzy point is the fuzzy identity group.

Theorem 8. The fuzzy fundamental group of the minimal fuzzy retraction of the n-dimensional fuzzy manifold $\tilde{F}^n$ homeomorphic to $\tilde{M}^n$ is the fuzzy identity group.

Proof. Let $\tilde{r}_i : \{\tilde{F}^n - (\tilde{B}_j^n)\} \rightarrow \tilde{F}^{n-1}$ be the fuzzy retractions map. Then, we have the following chains

$$\{\tilde{F}^n - (\tilde{B}_j^n)\} \xrightarrow{\tilde{r}_1^1} \{\tilde{F}_1^n - (\tilde{B}_1^n)\} \xrightarrow{\tilde{r}_2^1} \{\tilde{F}_2^n - (\tilde{B}_2^n)\} \rightarrow \ldots \{\tilde{F}_{n-1}^{\;n} - (\tilde{B}_{n-1}^n)\} \xrightarrow[i\to\infty]{lim\ \tilde{r}_i^1} \tilde{F}^{n-1},$$

$$\{\tilde{F}^{n-1} - (\tilde{B}_j^{n-1})\} \xrightarrow{\tilde{r}_1^2} \{\tilde{F}_1^{n-1} - (\tilde{B}_1^{n-1})\} \xrightarrow{\tilde{r}_2^2} \{\tilde{F}_2^{n-1} - (\tilde{B}_2^{n-1})\} \rightarrow \ldots \{\tilde{F}_{n-1}^{\;n-1} - (\tilde{B}_{n-1}^{n-1})\} \xrightarrow[i\to\infty]{lim\ \tilde{r}_i^2} \tilde{F}^{n-2}, \ldots,$$

$$\{\tilde{F}^{1} - (\tilde{B}_j^{1})\} \xrightarrow{\tilde{r}_1^n} \{\tilde{F}_1^{1} - (\tilde{B}_1^{1})\} \xrightarrow{\tilde{r}_2^n} \{\tilde{F}_2^{1} - (\tilde{B}_2^{1})\} \rightarrow \ldots \{\tilde{F}_{n-1}^{\;1} - (\tilde{B}_{n-1}^{1})\} \xrightarrow[i\to\infty]{lim\ \tilde{r}_i^n} \tilde{F}^{0}.$$

Thus the minimal fuzzy retractions of the n-dimensional fuzzy manifold $\tilde{F}^n$ coincide with the zero-dimensional fuzzy space which is the limit of fuzzy retractions. Thus, it is a fuzzy point and the fundamental group of a fuzzy point is the fuzzy identity group.

Theorem 9. Let $\tilde{S}_1^2 \subset \tilde{M}^4$ be a fuzzy deformation retract of $\tilde{M}^4$ and $\tilde{F} : \tilde{M}^4 \rightarrow \tilde{M}^4$ be a regular fuzzy folding, then $\bar{\tilde{F}} \circ \tilde{D}.\tilde{R} = \overline{\tilde{D}.\tilde{R}} \circ \tilde{F}$ and $\pi_1(\tilde{F}(\tilde{M}^4)) \cong \pi_1(\tilde{F}(\tilde{S}_1^2 \subset \tilde{M}^4))$.

Proof. Let the fuzzy deformation retract of $\tilde{M}^4$ be defined as

$$\tilde{\xi} : (\tilde{M}^4 - \{\tilde{\mu}_i\}) \times I \rightarrow (\tilde{M}^4 - \{\tilde{\mu}_i\}),$$

$$\tilde{\xi}(m,c) = (\frac{1-c}{1+c})\{(\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\cos(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}), \frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\sin(\frac{s_3}{1-ir(\eta)\sin\theta(\eta)}), \frac{s_1}{1-i}\cos(\frac{s_2}{1-ir(\eta)}), \frac{s_4}{1-\sqrt{p-1}}\} + (\frac{2c}{1+c})\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}), 0, (\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}), (\frac{s_4}{1-\sqrt{p-1}})\}$$

$$\tilde{\xi}(m,1) = \{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}), 0, (\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}), (\frac{s_4}{1-\sqrt{p-1}})\} = \tilde{S}_1^2 \subset \tilde{M}^4$$

and as a fuzzy deformation retract of $\tilde{M}^4$.

Now consider the fuzzy folding be defined as

$$\tilde{F} : \tilde{M}^4 \rightarrow \tilde{M}^4, \tilde{F} : (\tilde{S}_1^2 \subset \tilde{M}^4) \rightarrow (\tilde{S}_1^2 \subset \tilde{M}^4)$$

,such as $\tilde{F}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (|\tilde{x}_1|, |\tilde{x}_2|, |\tilde{x}_3|, |\tilde{x}_4|)$,

$$\tilde{F}\{\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)}), 0, (\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)}), (\frac{s_4}{1-\sqrt{p-1}})\} = \{\left|\frac{s_1}{1-i}\sin(\frac{s_2}{1-ir(\eta)})\right|, 0, \left|(\frac{s_1}{1-i})\cos(\frac{s_2}{1-ir(\eta)})\right|, \left|\frac{s_4}{1-\sqrt{p-1}}\right|\}.$$

Therefore $\bar{\tilde{F}} \circ \tilde{D}.\tilde{R} = \overline{\tilde{D}.\tilde{R}} \circ \tilde{F}$

and $\pi_1(\tilde{F}(\tilde{M}^4)) \cong \pi_1(\tilde{F}(\tilde{S}_1^2 \subset \tilde{M}^4))$.

Applications

(i) There are many applications to the density functions. In the field of medicine, for instance, we find that the patient who suffers from cancer which leads to many other sub-diseases, in the heart, kidney, blood, prostat, skin and nearly in all organs. That, who is attacked by the hepatitic virus C , in his body suffers from many symptoms as a result of his liver troubles. Also, in the field of medicine, the duzzeling of the eyes…. The eye can see fish in the air and some real figures in the air if the person concerned is un will, the phenomen represented by some one of the matrices g_1, g_0, g_2 where g_1 inside the cone, g_0 on the come, g_2 is outside the cone.

Mathematically, these factors and symptoms are expressed by the density functions where $(C(\lambda_1, \lambda_2, \ldots, \lambda_n), \mu_1, \mu_2, \ldots, \mu_n), \mu_1 \in [0,1]$, $\mu_2, \mu_3, \ldots, \mu_n \in [-1,1]$ the effective functions [12].

(ii) The geometric metric function represents real phenomena if the metric is a Riemannian metric, but in the case of a pseudo-Riemannian metric it will be applied in phenomena which are not real like the complex potential function and complex radiation[6-9].

(iii)- The Ritz variational method [1,11] during the calculation of the ground – state energy in a fuzzy framework. Consider a Hamilton *H*, and an arbitrary square integrable function Ψ , so that $\langle \Psi / \Psi \rangle = 1$. Considering Ψ as a fuzzy function and the ranking system as defined in [1,11], similar to [1,11] it can be shown that $\langle \psi / H / \psi \rangle$ is a fuzzy upper bound on E_0 (ground-stat energy). Now $\langle \psi / H / \psi \rangle$ should be minimizing the distance between E_0 and respect to a number of parameters $(\alpha_1, \alpha_2, \ldots)$. This can be done by minimizing distance between E_0 $\langle \psi / H / \psi \rangle$. The rest of the discussion is the same as that provided in [1,11].

Conclusion

In this paper the relations between fuzzy folding, fuzzy retraction, fuzzy deformation retract and fuzzy fundamental groups of $\tilde{M}^4$ are deduced. The connection between the limits of the fuzzy foldings and the fuzzy fundamental groups are obtained. Some applications are presented.

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