A study of Pseudolinear functions with convex optimization

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Abstract: In this paper we introduced Pseudolinear functions as a generalization of convex functions
Rep Opinion 2013;5(1):42-44]. (ISSN:1553-9873). http://www.sciencepub.net/report. 7

Keywords: Pseudolinear functions, ρ-convex functions ,convex optimization, Concave Optimization, non linear programming

Introduction

Pseudolinear functions were defined by **(5)** as functions which are both pseudoconvex and pseudoconcave. The following example illustrates the fact that if f and g are two pseudolinear functions with respect to same proportional function ρ, then f/g is not necessarily pseudolinear with respect to same proportional function p.

Example 1 : The real valued functions f and g defined on]0, 1[by

$$f(x) = (7x+3)/(2x+5)$$

$$g(x) = (9x+4)/(2x+5)$$

are pseudolinear with respect to same proportional function $p(x,u) = (2u+5)/(2x+5)$. But the function $f(x)/g(x) = (7x+3)/(9x+4)$ defined on]0, 1[is not pseudolinear with respect to proportional function p(x, u) because for x = 1/2, u = 1/4

$$f(x)/g(x) \neq f(u)/g(u) + p(x,u)\,(x-u)\,\nabla\,(f(u)/g(u)).$$

The following result illustrates that f/g is, however, pseudolinear with respect to a different proportional function.

Theorem 1 : If f and g are two pseudolinear functions defined on an open convex subset X of R^n with the same proportional function p(x, u) and g(x) > 0 for every x in X, then f/g is also pseudolinear on X with respect to proportional function $\overline{p}(x,u) = p(x,u)\,g(u)/g(x)$.

Proof : Since f and g are pseudolinear functions with respect to same proportional function p it follows that for x, u in X

$$f(x) = f(u) + p(x,u)\,(x-u)^T\,\nabla f(u)$$

$$g(x) = g(u) + p(x,u)\,(x-u)^T\,\nabla g(u)$$

It can be shown that

$$p(x,u)\,(x-u)^T\,\nabla(f(u)/g(u)) = g(x)[(f(x)/g(x)) - (f(u)/g(u))]/g(u)$$

Thus,

$$f(x)/g(x) - f(u)/g(u) = p(x,u)\;\,g(u)(x-u)^T\,\nabla(f(u)/g(u))/g(x)$$

which implies that f/g is pseudolinear with respect to proportional function $\overline{p}(x,u) = p(x,u)\,g(u)/g(x)$.

Remark 1 : In the example considered above, the function f/g is pseudolinear with respect to the proportional function

$$\overline{p}(x,u) = p(x,u)\,g(u)/g(x)$$
$$= (9u+4)/(9x+4)$$

The class of pseudolinear functions is generalized to a new class of functions called η-pseudolinear functions. Let $f: X \to R$, $p: X \times X \to R$, $\eta: X \times X \to R^n$, where X is an open subset of R^n.

Definition 1 : The function f is said to be η-pseudolinear if there exist functions p(x, u) and $\eta(x, u)$, such that, p(x, u) > 0 for $x, u \in X$ and

$$f(x) = f(u) + p(x, u)\,\eta(x, u)^T\,\nabla f(u)$$

The following theorem follows on the lines of Theorem 1.

Theorem 2 : If f and g are two η-pseudolinear functions defined on an open subset X of R^n with same proportional function $p(x, u)$ and $g(x) > 0$ for every x in X, then f/g is η-pseudolinear on X with respect to new proportional function $\overline{p}(x, u) = p(x, u)\,g(u) / g(x)$.

The following theorems establish certain sufficient conditions for composite functions to be η-pseudolinear.

Theorem 3 : Let $\phi : R^n \to R^n$ be a surjective function with $\nabla\phi(x)$ onto for each $x \in R^n$ and $f_i : R^n \to R, i = 1, 2, ..., k$ be pseudolinear functions with respect to proportional function p_i, then the function $h : R^n \to R^n$ defined by

$$h(x) = (f_1(\phi(x)),\ f_2(\phi(x))......, f_k(\phi(x)))$$

is η-pseudolinear.

Proof : Let $x, u \in R^n$. Let $w = \phi(x),\ \ z = \phi(u)$. We have

$$f_i(\phi(x)) - f_i(\phi(u)) = f_i(w) - f_i(z)$$
$$= p_i(w, z)\,(w - z)^T\,\nabla f_i(z)$$

and f_i is pseudolinear with respect to proportional function p_i, i = 1, 2, …, k. Since $\nabla\phi(u)$ is onto, the equation $w - z = \nabla\phi(u)^T\ \eta(x,)$ is solvable. Thus, we get

$$f_i(\phi(x)) - f_i(\phi(u)) = p_i(w, z)\,\eta(x, u)^T\,\nabla\phi(u)\,\nabla f_i(z)$$
$$= p_i(w, z)\,\eta(x, u)^T\,\nabla(f_i \circ \phi)\,(u)$$
$$= p_i(\phi(x), \phi(u))\,\eta(x, u)^T\,\nabla(f_i \circ \phi)(u)$$
$$= \overline{p}_i(x, u)\,\eta(x, u)^T\,\nabla(f_i \circ \phi)\,(u)$$

where $\overline{p}_i(x, u) = p_i(\phi(x), \phi(u))$. Since each component of h is η-pseudolinear, it follows that h is η-pseudolinear.

Theorem 4 : Let $g : R^n \to R$ be continuity differentiable η-pseudolinear with respect to proportional function q and $f : R \to R$ be pseudolinear with respect to proportional function p. Then $(f \circ g)\,(x)$ is η-pseudolinear with respect to new proportional function.

Proof : Let $x, u \in R^n$. Let $w = g(x),\ \ z = g(u)$.

$$f(g(x)) - f(g(u)) = f(w) - f(z)$$
$$= p(w, z)\,(w - z)\,\nabla f(z) \qquad (1)$$

as f is pseudolinear with respect to p. Also

$$w - z = g(x) - g(u)$$
$$= q(x, u)\,\eta(x, u)^T\,\nabla g(u)$$

as g is η-pseudolinear with respect to q. Substituting the value of $w - z$ in (1), we get

$$f(g(x)) - f(g(u)) = p(w, z)\,q(x, u)\,\eta(x, u)^T\,\nabla g(u)\,\nabla f(z)$$
$$= p(g(x), g(u))\,q(x, u)\,\eta(x, u)^T\,\nabla(f \circ g)\,(u)$$

$$= r(x,u)\,\eta(x,u)^T\,\nabla(f \circ g)(u)$$

where $r(x,u) = p(g(x), g(u))\,q(x,u)$. Thus it follows that $(f \circ g)(x)$ is η-pseudolinear with respect to r.

We now define second order pseudolinear twice differentiable functions. Let $f: X \to R$ be a twice differentiable function defined on a non-empty open subset X of R^n. Let $p: X \times X \to R^n$, $q: X \times X \to R$.

Definition 2 : The function f is said to be second order pseudolinear at $u \in X$ with proportional function q if there exist functions p(x, u), q(x, u) such that q(x, u) > 0 and for $x \in X$

$$f(x) - f(u) + \frac{1}{2}p^T\nabla^2 f(u)p = q(x,u)(x-u)^T(\nabla f(u) + \nabla^2 f(u)p)$$

Remark 2: Every second order pseudolinear function is both second order pseudoconvex and second order quasiconvex.

Second order η-pseudolinear functions are defined as an extension of η-pseudolinear functions and second order pseudolinear functions. Let $\eta: X \times X \to R^n$.

Definition 3 : The function f is said to be second order η-pseudolinear at $u \in X$ with proportional function q if there exist functions $p(x,u), q(x,u)$ and $\eta(x,u)$ such that q(x, u) > 0 and for $x \in X$

$$f(x) - f(u) + \frac{1}{2}p^T\nabla^2 f(u)p = q(x,u)\,\eta(x,u)^T\,(\nabla f(u) + \nabla^2 f(u)p)$$

Conclusion

In the above examples it is concluded that if *f* and *g* are two pseudolinear functions with respect to same proportional function ρ, then f/g is not necessarily pseudolinear with respect to same proportional function p.

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1/7/2013