Solution of Volterra Integral Equation of Second Kind

Vijendra S Rawat & Mayank Pawar

Department of Mathematics, Teerthankar Mahaveer university, Moradabad. UP India.
E-mail: vijendraprof@gmail.com

Abstract:-In this research paper we are examine the solution of volterra integral equation of second kind.
[**Vijendra S Rawat & Mayank Pawar. Solution of Volterra Integral Equation of Second Kind**. Researcher. 2010;2(11):52-55]. (ISSN: 1553-9865). http://www.sciencepub.net.

Key Words: Numerical Integration, Gaussian Integration, optimization theory, Non linear VIE.

1.1 Introduction

There are many different type of integral equations which also include Volterra and fredholm ones of second and first kind. A Volterra integral equation (VIE) of the second kind has the form

$$v(x) = \psi(x) + \int_a^x K(x,t,v(t))dt, a \le x \le X, \tag{1.1}$$

and a Fredholm of integral equation (FIE)of the second kind has the form

$$v(x) = \psi(x) + \int_a^b K(x,t,v(t))dt, a \le x \le b.$$

The kernel $K(x,t,v(t))$ in both cases is either continuous in all its three variable, or weakly singular for example of the form

$$K(x,t,v(t)) = \frac{H(x,t,v(t))}{|x-t|^{\beta}}, 0 < \beta < 1, \tag{1.2}$$

Where $H(x,t,v(t))$ is continuous in all its three variables.

VIEs of the second kind in the form (1.1) with weakly singular kernels of the form (1.2):

$$v(x) = \psi(x) + \int_0^x K(x,t,v(t))dt, 0 \le t \le x \le X, \tag{1.3}$$

With

$$K(x,t,v(t)) = \frac{H(x,t,v(t))}{|x-t|^{\beta}}, 0 < \alpha < 1, 0 \le t \le x \le X, \tag{1.4}$$

And

$$H(x,t,v(t)) = cv(t) \tag{1.5}$$

Where C is a constant.

This is, we are considering VIEs of the form:

$$v(x) = \psi(x) + c\int_0^x \frac{v(t)}{\sqrt{x-t}}dt, \tag{1.6}$$

In the order to solve these equation we will make we will make use of generalized Newton-cotes quadrature rules ([6],p. 47, [3], p. 864). Comparisons are made with a numerical approach using the conversion to ODEs concept by Abdalkhani ([1]).

Notation preliminaries: In all methods we consider a mesh of the form:

$$0 = x_0 < x_1 < x_2 < \ldots\ldots. < x_n = X \tag{1.7}$$

The stepsize is defined $h_i = x_{i+1} - x_i, i = 1, 2, 3 \ldots\ldots, n$.

1.2 Generalized Newton Cotes

If we consider VIEs with weakly singular kernels of the form:

$$v(x) = \psi(x) + c\int_0^x \frac{v(t)}{\sqrt{x-t}}dt, \tag{1.8}$$

And discretise at $x = x_i$ given by (1.7), we have that:

$$v(x_i) = \psi(x_i) + c\int_0^{x_i} \frac{v(t)}{\sqrt{x_i-t}}dt \Rightarrow \tag{1.9}$$

$$v(x_i) = \psi(x_i) + c\sum_{j=1}^{i-1}\int_{x_j}^{x_{j+1}} \frac{v(t)}{\sqrt{x_i-t}}dt \tag{1.10}$$

Using a lagrange interpolating polynomial we approximate the u(t) inside the integral with $l_0^j(t)v_j + l_1^j(t)v_{j+1}$ so we get:

$$v(x_i) = \psi(x_i) + c\sum_{j=1}^{i-1}\int_{x_j}^{x_{j+1}} \frac{l_0^j(t)v_j + l_1^j(t)v_{j+1}}{\sqrt{x_i-t}}dt \Leftrightarrow \tag{1.11}$$

$$v(x_i) = \psi(x_i) + c\sum_{j=1}^{i-1}\left\{v_j\int_{x_j}^{x_{j+1}} \frac{l_0^j(t)}{\sqrt{x_i-t}}dt + v_{j+1}\int_{x_j}^{x_{j+1}} \frac{l_1^j(t)}{\sqrt{x_i-t}}dt\right\} \tag{1.12}$$

Or

$$v_i\left\{1 - c\int_{x_{i-1}}^{x_i} \frac{l_1^{i-1}(t)}{\sqrt{x_i-t}}dt\right\}$$
$$= \psi(x_i) + c\sum_{j=1}^{i-1} v_j\int_{x_j}^{x_{j+1}} \frac{l_0^j(t)}{\sqrt{x_i-t}}dt + c\sum_{j=1}^{i-2} v_{j+1}\int_{x_j}^{x_{j+1}} \frac{l_1^j(t)}{\sqrt{x_i-t}}dt \quad . \tag{1.13}$$

We have that

$$l_0^j(t) = \frac{t - x_{j+1}}{x_j - x_{j+1}} \tag{1.14}$$

$$l_1^j(t) = \frac{t - x_j}{x_{j+1} - x_j} \tag{1.15}$$

And we also have

$$I_1(Z, A, B) = \int_A^B \frac{dt}{\sqrt{Z-t}}, \tag{1.16}$$

$$I_2(Z, A, B) = \int_A^B \frac{t\,dt}{\sqrt{Z - t}}, \quad (1.17)$$

With the help of these integrals we can rewrite the above equation in order to compute the desire solution : for example:

$$\int_{x_j}^{x_{j+1}} \frac{l_0^j(t)}{\sqrt{x_i - t}} dt \; \frac{1}{x_j - x_{j+1}} \int_{x_j}^{x_{j+1}} \frac{t - x_{j+1}}{\sqrt{x_i - t}} dt$$

$$= \frac{1}{x_j - x_{j+1}} \left\{ \int_{x_j}^{x_{j+1}} \frac{t}{\sqrt{x_i - t}} dt - x_{j+1} \int_{x_j}^{x_{j+1}} \frac{dt}{\sqrt{x_i - t}} \right\} \Rightarrow$$

$$\int_{x_j}^{x_{j+1}} \frac{l_0^j(t)}{\sqrt{x_i - t}} dt = \frac{1}{x_j - x_{j+1}} \left\{ I_2(x_i, x_j, x_{i+1}) - x_{j+1} - I_1(x_i, x_j, x_{i+1}) \right\}. \quad (1.18)$$

1.3 A numerical approach using interpolating polynomials based on Abdalkhani ([1])

if in (1.4)we replace $(x-t)^{-\beta}$ by a polynomial of degree N in x and t, $P_{N,a}(x-t)$ then (1.3) becomes $v(x) = \Psi(x) + \int_0^x P_{N,a}(x-t) H(x, t, v(t))dt, .$ (1.19)

Theorem1 (Abdalkhani, [1], p. 251)

If we approximate $(x-t)^{-\beta}$ by $P_{N,\beta}(x-t)$ given by

$$P_{N,\beta}(x-t) = \frac{2\Gamma\left(\frac{3}{2} - a\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(3-a)} \sum_{n=0}^{N} \frac{(n+1)(a)_n}{(3-a)_n} U_n(1 - 2x + 2t), \quad (1.20)$$

For (x,t) ∈ S ,where $S = \{(x,t) : 0 \le t \le x \le X\}$ and U_n are the chebychev polynomials of the second kind and $(a)_n$ is defined by

$$(a)_n = \begin{cases} 1 & if\, n = 0 \\ a(a+1)(a+2)....(a+n-1) & if\, n = 1\ 2\ 3\ ...., \end{cases} \quad (1.21)$$

Then for $x \in [o, X]$. We have

$$\int_0^t [(x-t)^{\beta} - P_{N,\beta}(x-t)]dt = O\left(N^{-2(1-\beta)}\right). \quad (1.22)$$

Chebychev polynomials of the second kind are given by the following explicit expression (cf. [16], p. 29)

$$U_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(m-n)!}{m!(n-2m)!} (2x)^{n-2m} \quad (1.23)$$

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (1.24)$$

Theorem 2 (Abdalkhani ([1], P. 250)

Assume that (1.3) and (1.19) possess, respectively unique solution $v \in C(I)$ and $W \in C(I)$,and suppose that

$$\left|\int_0^x ((x-t)^{-\beta} - P_{N,\beta}(x-t))\,dt\right| < \in_1 \text{, for all } 0 \le t \le x \le X. \tag{1.25}$$

Let $\hat{W}(x)$ be any numerical approximation to w(x) such that $\left|W(x) - \hat{W}(x)\right| \le \in_2$ for all x, $0 \le x \le X$.

In addition, let $K(x,t,v)$ be continuous in the region

$$\Omega = \{(x,t,v) : (x,t) \in S \text{ and } |u - \psi(x)| \le B\}. \tag{1.26}$$

Also let $|K(x,t,v) - K(a,t,u)| \le L|v-u|$. Then

$$\left|v(x) - \hat{W}\right| \le C_1 \in_1 + C_2 \in_2, \tag{1.27}$$

Where C1 and C2 are real constants.

References:-

[1] Abdalkahani, J. (1990). A Numerical Approach to the solution of Abel integral equation of the second kind with the nonsmooth solution. Journal of Computational and Applied Mathematics, 29,249-255.
[2] Allen, E. S. (1941). The Scientific work of Vito Volterra. American Mathematical Monthly, 48,516-519.
[3] Baker, C.T.H.(1977). The Numerical Solution of Integral Equations. Oxford: Clarendon Press.
[4] Brunner, H. (2004). Collocation Methods for Volterra Integral and Related Functional Equation. Cambridge: University Press.
[5] Davis, P. J. and Rabinowitz, P. (1975). Method of Numerical Integration. New York: Academic Press.
[6] Linz, P. (1985). Analytical and Numerical Method for Volterra Equation. Philadelphia: SIAM.

10/4/2010