Duals of Riesz bases in finitely or countably generated Hilbert C*-modules

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Abstract: We investigate frames, Riesz bases and their dual modular frames in Hilbert C*-modules. A Riesz basis for a Hilbert C*-modules can have many different dual modular frames which are not necessarily Riesz bases and they always are not unique. We characterize those Riesz basis that have unique duals. Finally, we obtain a necessary and suffieint condition for a dual of a Riesz bases to be again a Riesz bases.

[Azita Barani. **Duals of Riesz bases in finitely or countably generated Hilbert C*-modules..** *Researcher* 2014;6(7):11-18]. (ISSN:1553-9865). http://www.sciencepub.net/researcher2

Keywords: Hilbert C*-modules, Frame, Riesz bases, Dual modular frames

1. Introduction

Frames in Hilbert spaces introduced by Gabor and they serve as a replacement for bases, but with more flexibility. Later frames were originally introduced by Duffin and Schaffer[2] to deal with some problems in non-harmonic Fourier analysis. Hilbert C*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C*-algebra rather than in the field of complex numbers [4]. These frames are called *Hilbert* C*-*modular frames* or just simply *modular frames*. These concept are generalizations of some results in[7].

Riesz bases play important roles in the study of Hilbert space frame theory (cf.[6]). However, we will encounter several obstacles when we deal with Riesz bases and frames for Hilbert C*-modules. A Riesz basis always has a canonical dual which is ecessarily a Riesz basis. In this paper we obtain a necessary and sufficient condition for a dual of a Riesz basis to be again a Riesz basis. Inparticular, we characterize those modular Riesz bases thatave unique duals. The characterization is given in terms of the properties of the range spaces of the *analysis operators*. As a consequence, we show that when the underlying C*-algebra is commutative, every modular Riesz basis has a unique Riesz dual, although it maybe have other duals that are not Riesz basis.(see Example 4.2)

2.Perliminaries

We first recall definitions and results about Hilbert C*-modules, frames and Riesz basis in Hilbert C*-modules.

In this paper *N* will denote the set of natural numbers and *J* will be a finite or countable subset of *N*.

Definition 2.1. Let *A* be a (unital) C*-algebra and *H* be a (left)*A*-module. Suppose that The linear structures given on *A* and *H* are compatible, i.e. $\lambda(ax)=a(\lambda x)$ for every $\lambda \in C, a \in A$ and $x \in H$. Assume that there exists a mapping $\langle .,. \rangle : H \times H \to A$ with the properties: $(i)\langle x,x \rangle \geq 0$ for every $x \in H$, $(ii)\langle x,x \rangle = 0$ if and only if $x=0$, $(iii)\langle x,y \rangle = \langle y,x \rangle^*$ for every $x,y \in H$, $(iv)\langle ax,y \rangle = a\langle x,y \rangle$ for every $x,y \in H$, and every $a \in A$, $(v)\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$ for every $x,y,z \in H$ .Then the pair $\{H,\langle .,. \rangle\}$ is called a (left)pre-Hilbert *A*-module. The map $\langle .,. \rangle$ is said to be an *A*-valued inner product. If the pre-Hilbert *A*-module $\{H,\langle .,. \rangle\}$ is complete with respect to the induced norm $\|x\| = \|\langle x,x \rangle\|^{\frac{1}{2}}$ then it is called a Hilbert *A*-module.

In this paper we focus on finitely and countably generated Hilbert A-module over unital C*-algebra A. A Hilbert A-module H is *(algebraically) finitely generated* if there exists a finite set $\{x_1, ..., x_n\} \subseteq H$ such that every element $x \in H$ can be expressed as an A-linear combination $x = \sum_{j=1}^{j=n} a_j x_j$, $a_j \in A$. A Hilbert A-module H is *countably generated* if there exists a countable set $\{x_j\}_{j\in N} \subseteq H$ equals the norm-closure of the linear span (over C and A) of this set.

Definition 2.2.(see [4]) Let A be a unital C^* – algebra and J be a finite or countable index set. A (finite or countable) sequence $\{x_j\}_{j\in J}$ of elements in a Hilbert A-module H is said to be a *frame* for H if there exist two constants $C, D \rangle 0$ such that the frame inequality

$$\textbf{(1)} C\langle x, x\rangle \le \sum_{j\in J}\langle x, x_j\rangle\langle x_j, x\rangle \le D\langle x, x\rangle$$

holds for every $x \in H$, where the sum in the middle of the inequality (1) is convergent in norm. The numbers C and D are called *frame bounds*. The sequence $\{x_j\}_{j\in J}$ is called a (standard) *Bessel sequence* with Bessel bound D if there exists D>0 such that

$$\sum_{j\in J}\langle x, x_j\rangle\langle x_j, x\rangle \le D\langle x, x\rangle$$

The sequence $\{x_j\}_{j\in J}$ satisfies the lower frame bound if there exists a C>0

$C\langle x, x\rangle \le \sum_{j\in J}\langle x, x_j\rangle\langle x_j, x\rangle$ if we only require

The frame $\{x_j\}_{j\in J}$ is saide to be a *tight frame* if C=D, and said to be *normalized* if C=D=1.

We consider standard (normalized tight) frames on finitely or countably generated Hilbert A-module over unital C*-algebra A.

Definition 2.3.(see [4]) A frame $\{x_j\}_{j\in J}$ for a Hilbert A-module H is said to be a (*standard*) *Riesz basis* for H if it satisfies:

(i) $x_j \neq 0$ for all j;

(ii) if an A-linear combination $\sum_{j\in S} a_j x_j$ with coeffcients $\{a_j : j \in S\} \subseteq A$ and $S \subseteq J$ is equal to zero, then every summand $a_j x_j$ is equal to zero.

3. Dual sequences of frames and Riesz basis

For a unital C*-algebra A, let $\ell^2(A)$ be the Hilbert A-module, see[4], define by

$$\ell^2(A) = \left\{\{a_j\}_{j\in J} \subset A : \sum a_j a_j^* \textit{ converges in } \|.\|\right\}$$

For any standard frame $\{x_j\}_{j\in J}$ of a finitely or countably generated Hilbert A-module H, the frame transform of the frame $\{x_j\}_{j\in J}$ is definded to be the map

$$\theta(x): H \to \ell^2(A) \quad , \quad \theta(x) = \{\langle x, x_j\rangle\}_{j\in J}$$

that is bounded, A-linear and adjointable with adjoint

$$\theta^*(x): \ell^2(A) \to H \quad , \quad \theta^*(e_j) = x_j$$

for a standard basis $\{e_j\}_{j\in J}$ of the Hilbert A-module $\ell^2(A)$ and all $j \in J$. See[3]

Moreover for every $x \in H$,

$$|\theta(x)|^2 = \langle\theta(x), \theta(x)\rangle = \sum_{j\in J}\langle x, x_j\rangle\langle x_j, x\rangle$$

Therefore θ is one-to-one with a closed range which is complemented in $\ell^2(A)$,

$\ell^2(A) = \theta(H) + Ker\theta^*(x)$.

We note that $\theta^{*}\left|\theta(H)\right.$ is an invertible operator and the frame operator $S=\left(\theta^{*}\theta\right)^{-1}$ is a positive invertible bounded operator on H such that for every $x\in H$,

$$(2)\quad x=\sum_{j\in J}\left\langle x,Sx_j\right\rangle x_j=\sum_{j\in J}\left\langle x,x_j\right\rangle Sx_j$$

The sequence $\left\{Sx_j\right\}_{j\in J}$ is a frame for H and is called the *canonical dual frame* of $\left\{x_j\right\}_{j\in J}$.

Now suppose that $\left\{x_j\right\}_{j\in J}$ is a Bessel sequence of a finitely or countably generated Hilbert A-module H, the associated *analysis operator* $T_X:H\to\ell^2(A)$ is defined by

$$T_X x=\sum_{j\in J}\left\langle x,x_j\right\rangle e_j \qquad x\in H.$$

Note that analysis operator T_X is adjointable and adjoint T_X^{*} fulfills $T_X^{*}e_j=x_j$ for all j.

Definition 3.1. Suppose that H is a Hilbert A-module over a unital C*-algebra A. Let $\left\{x_j\right\}_{j\in J}$ be a (standard) frame and $\left\{y_j\right\}_{j\in J}$ a sequence of H. Then $\left\{y_j\right\}_{j\in J}$ is said to be a (standard)*dual sequence* of $\left\{x_j\right\}_{j\in J}$ if

$$(3)\qquad x=\sum_{j\in J}\left\langle x,y_j\right\rangle x_j$$

Holds for all $x\in H$, where the sum in (3) converges in norm. The pair $\left\{x_j\right\}_{j\in J}$ and $\left\{y_j\right\}_{j\in J}$ are called a *dual frame pair* when $\left\{y_j\right\}_{j\in J}$ is also a frame.

Definition 3.2. P_n be a projection on $\ell^2(A)$ that maps each element to its nth component, i,e.

$P_n x=\left\{u_j\right\}_{j\in J}$, where

$$u_j=\begin{cases}x_n & if \quad j=n\\ 0 & if \quad j\neq n\end{cases}$$

For each $x=\left\{x_j\right\}_{j\in J}\in\ell^2(A)$.

Theorem 3.3.(see[7])Let $\left\{x_j\right\}_{j\in J}$ be a frame of a finitely or countably generated Hilbert A-module H over a unital C*-algebra A. then $\left\{x_j\right\}_{j\in J}$ is a riesz basis if and only if $x_j\neq o$ and $P_n\left(Rang\left(T_X\right)\right)\subseteq\left(Rang\left(T_X\right)\right)$ for all $j\in J$, where T_X is the analysis operator of $\left\{x_j\right\}_{j\in J}$.

Corollary 3.4. Suppose that $\left\{x_j\right\}_{j\in J}$ it is a frame of finitely or countably generated Hilbert A-module H, then $\left\{x_j\right\}_{j\in J}$ is a Riesz basis if only if

(i) $x_j\neq o$ for each $j\in J$;

(ii) if $\sum_{j\in J}c_j x_j=0$ for some sequence $\left\{c_j\right\}_{j\in J}\in\ell^2(A)$, then $c_j x_j=0$ for each $j\in J$.

Proof. Suppose first that $\left\{x_j\right\}_{j\in J}$ is a Riesz basis. Then obvious hold conditions (i) and (ii).

Now suppose that $\left\{x_j\right\}_{j\in J}$ is a frame in H with conditions (i) and (ii). Then $\sum_{j\in J}c_j x_j=0$ for some sequence $\left\{c_j\right\}_{j\in J}\in\ell^2(A)$, then $c_j x_j=0$ for each $j\in J$. Therefore we have for , $y_n\in H$,

$$0=\left\langle y_n, \sum_{j\in J} c_j x_j \right\rangle = \sum_{j\in J}\left\langle y_n, c_j x_j\right\rangle$$
$$=\sum_{j\in J}\left\langle y_n, x_j\right\rangle c_j^* = \left\langle x, x_n\right\rangle c_j^* = \left\langle x, c_n x_n\right\rangle$$

that we get

$$\left\langle y_n, x_j\right\rangle = \begin{cases} \left\langle x, x_n\right\rangle & j=n \\ 0 & j\neq n \end{cases}$$

Then there exists a $y_n \in H$ such that $T_X y_n = P_n T_X x$, which this implise that $P_n T_X x \in Rang\left(T_X\right)$ therefore $P_n\left(Rang\left(T_X\right)\right)\left(T_X\right)$ then by Theorem3.2 $\{x_j\}_{j\in J}$ is a Riesz basis.

We end up this section with the following theorems which will be used in the proof of theorem 4.3.

Theorem 3.5.(see[7]) Suppose that H is a finitely or countably generated Hilbert A-module over a unital C*-algebra A. Let $\{x_j\}_{j\in J}$ be a frame of H with analysis operator T_X, then the following statements are equinalent.

$(i)\ \{x_j\}_{j\in J}$ has a unique dual frame;

$(ii)\ Rang\left(T_X\right)=\ell^2(A)$;

(iii) if $\sum_{j\in J} c_j x_j = 0$ for some sequence $\{c_j\}_{j\in J} \in \ell^2(A)$, then $c_j = 0$ for each $j\in J$.

In case the equinalent conditions are satisfied, $\{x_j\}_{j\in J}$ is a Riesz basis.

Theorem 3.6. (see[7]) Suppose that $\{x_j\}_{j\in J}$ is a Riesz basis of finitely or countably generated Hilbert A-module H over a unital C*-algebra A. Let $\{y_j\}_{j\in J}$ be a sequence of H**.** Then the following statements are equinalent.

$(i)\ \{y_j\}_{j\in J}$ is a dual frame of $\{x_j\}_{j\in J}$;

$(ii)\ \{y_j\}_{j\in J}$ is a dual Bessel sequence of $\{x_j\}_{j\in J}$;

(iii) for each $j\in J$, $y_j = S^{-1}x_j + z_j$, where S is the frame operator of $\{x_j\}_{j\in J}$, and $\{z_j\}_{j\in J}$ is a Bessel sequence of H satisfying $\left\langle x, z_j\right\rangle x_j = 0$ for all $x\in H$ and $j\in J$.

4.Dual modular frames for Riesz bases

The aim of section is to have a detailed investigation on the dual sequences Riesz bases in Hilbert C*-modules**.**

The following example show that in Hilbert C*-modules the dual Riesz basis of Riesz basis is not unique.

Example 4.1. Let $A=M_{2\times 2}(C)$ denote the C*-algebra of all 2×2 complex matrices. Let $H=A$ and for any $A,B\in H$ define

$\left\langle A,B\right\rangle = AB^*$.

Then H is a Hilbert A-module.

Let $E_{i,j}$ be the 2×2 matrix with 1 in the *(i,j)* th entry and 0 elsewhere, where $1\le i,j\le 2$. Then $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of H and it is a dual of itself. One can check that $\{E_{1,1}+E_{2,1}, E_{2,2}\}$ is also a dual Riesz basis of $\{E_{1,1}, E_{2,2}\}$.

Note that even the dual sequence of a Riesz basis in Hilbert C*-modules is a Bessel sequence, but it is not a Riesz basis. We have the folloing example.

Example 4.2. Let ℓ^{∞} be the set of all bounded complex-valued sequence. For any $u=\{u_j\}_{j\in N}$ and $v=\{v_j\}_{j\in N}$ in ℓ^{∞}, we define

$uv=\{u_j v_j\}_{j\in N}$, $u^{*}=\{\overline{u_j}\}_{j\in N}$ and

$\|u\|=\max_{j\in N}|u_j|$,

Then $A=\{\ell^{\infty},\|.\|\}$ is a C*-algebra. Let $H=c_0$ be the set of all sequences converging to zero.

For any $u,v\in H$ we1 define

$\langle u,v\rangle=uv^{*}=\{u_j\overline{v_j}\}_{j\in N}$.

Then H is a Hilbert A- module.

Now let $x_j=e_j$ and

$$y_j=\begin{cases} e_1+e_2 & j=1,2 \\ e_j & j\neq 1,2\end{cases}$$

then $\{y_j\}_{j\in J}$ is a Bessel sequence, and satisfies $x=\sum_{j\in N}\langle x,y_j\rangle x_j$ for all $x\in H$. Therefore, by theorem 3.8 in [7], $\{y_j\}_{j\in J}$ is a frame of H. But obviously $\{y_j\}_{j\in J}$ is not a Riesz basis.

For the case of a dual sequence of a Riesz basis to be a Riesz basis, we have the following characterization.

Theorem 4.3. Let $\{x_j\}_{j\in J}$ be a Riesz basis and $\{y_j\}_{j\in J}$ a sequence of finitely or countably generated Hilbert A-module H over a unital C*-algebra A. Then $\{y_j\}_{j\in J}$ is a dual Riesz basis of $\{x_j\}_{j\in J}$ if and only if for each $j\in J$, $y_j=S^{-1}x_j+z_j$, where S is the frame operator of $\{x_j\}_{j\in J}$ and $\{z_j\}_{j\in J}$ is a Bessel sequence of H with the property that for each $j\in J$ there exists $b_j\in A$ such that $z_j=b_jS^{-1}x_j$ and $\langle x,x_j\rangle b_jx_j=0$ holds for all $x\in H$.

Proof. '$\Rightarrow$' Suppose first that $\{y_j\}_{j\in J}$ is a dual Riesz basis of $\{x_j\}_{j\in J}$ and let $z_j=y_j-S^{-1}x_j$. Then it is obvious that $\{z_j\}_{j\in J}$ is a Bessel sequence of H. Now fix an $n\in J$. From

$$y_n=\sum_{j\in J}\langle y_n,x_j\rangle y_j$$

we can ifer that

$y_n=\langle y_n,x_n\rangle y_n$ i.e.

$S^{-1}x_n+z_n=\langle S^{-1}x_n+z_n,x_n\rangle(S^{-1}x_n+z_n)$

Consequently, we have

$z_n=\langle z_n,x_n\rangle S^{-1}x_n$

$+\langle S^{=1}x_n,x_n\rangle z_n+\langle z_n,x_n\rangle z_n$.

To show that

$\langle S^{-1}x_n,x_n\rangle z_n+\langle z_n,x_n\rangle z_n=0$,

it sufficient to show that

$\langle S^{-1}x_n,x_n\rangle\langle z_n,x\rangle+\langle z_n,x_n\rangle\langle z_n,x\rangle=0$

holds for all $x\in H$. Note that

$$x=\sum_{j\in J}\langle x,y_j\rangle x_j=\sum_{j\in J}\langle x,S^{-1}x_j\rangle x_j$$
$$+\sum_{j\in J}\langle x,z_j\rangle x_j=x+\sum_{j\in J}\langle x,z_j\rangle x_j$$

Which implies that $\sum_{j\in J}\langle x,z_j\rangle x_j=0$ and so $\langle x,z_j\rangle x_j=0$ for all $x\in H$ and $j\in J$. Particularly, we have $\langle x,z_n\rangle x_n=0$ for all $x\in H$. This yields that $\langle x,z_n\rangle\langle x_n,z_n\rangle=0$ and $\langle x,z_n\rangle\langle x_n,S^{-1}x_n\rangle=0$.

Equivalently, $\langle z_n,x_n\rangle\langle z_n,x\rangle=0$ and $\langle S^{-1}x_n,x_n\rangle\langle z_n,x\rangle=0$.

Therefore $z_n = b_n S^{-1} x_n$,

where $b_n = \langle z_n, x_n \rangle$.

From $\langle x_n, z_n \rangle x_n = 0$, we have

$\langle y, x_n \rangle \langle x_n, z_n \rangle \langle x_n, x \rangle = 0$ for all $x, y \in H$,

which is equivalent to

$\langle x, x_n \rangle \langle z_n, x_n \rangle \langle x_n, y \rangle = 0$,

this implies that

$$\langle x, x_n \rangle b_n x_n = \langle x, x_n \rangle \langle z_n, x_n \rangle x_n = 0$$

'$\Leftarrow$' Suppose now that for each there $j \in J$ exists $b_j \in A$ such that $z_j = b_j S^{-1} x_j$ and $\langle x, x_j \rangle b_j x_j = 0$ holds for all $x \in H$.

Then for all $x, y \in H$ we have $\langle x, x_j \rangle b_j \langle x_j, y \rangle = 0.$

Equivalently,

$\langle y, x_j \rangle b_j^* \langle x_j, x \rangle = 0$.

This implies that

$\langle y, x_j \rangle b_j^* x_j = 0$ for all $y \in H$.

Now for arbitrary $x \in H$,

$$\sum_{j \in J} \langle x, y_j \rangle x_j = \sum_{j \in J} \langle x, S^{-1} x_j \rangle x_j + \sum_{j \in J} \langle x, z_j \rangle x_j = x + \sum_{j \in J} \langle x, S^{-1} x_j \rangle x_j$$
$$= x + \sum_{j \in J} \langle x, S^{-1} x_j \rangle b_j^* x_j = x + \sum_{j \in J} \langle S^{-1} x, x_j \rangle b_j^* x_j = x$$

Which implies that $\{y_j\}_{j \in J}$ is a dual sequence of $\{x_j\}_{j \in J}$, one can easily see that $\{y_j\}_{j \in J}$ is a Bessel sequence. Then by theorem 3.8 in [7], $\{y_j\}_{j \in J}$ is a dual Riesz basis of $\{x_j\}_{j \in J}$. To complete the proof, we need to show that $\{y_j\}_{j \in J}$ is a dual Riesz basis of *H*.

If $\sum_{j \in J} a_j y_j = 0$, then we have

$$0 = \sum_{j \in J} a_j \left(S^{-1} x_j + b_j S^{-1} x_j \right) = \sum_{j \in J} a_J \left(1 + b_j\right) S^{-1} x_j.$$

Therefore

$a_j \left(1 + b_j\right) S^{-1} x_j = 0$, i.e.

for $a_j x_j = 0$ all $j \in J$.

Now assume on the contrary that $y_n = 0$ for some $n \in J$.

Then

$z_n = -S^{-1} x_n$.

It follows that

$$0 = \langle x, x_n \rangle b_n x_n = \langle x, x_n \rangle S z_n = -\langle x, x_n \rangle x_n$$

holds for all $x \in H$. Let $x = S^{-1} x_n$, we have

$$0 = -\left\langle S^{-1} x_n, x_n \right\rangle x_n = -x_n,$$

and so $-x_n = 0$, a contradiction. Then $y_j \neq 0$ for each $j \in J$.

Note that under the conditions of following Corollary, though a Riesz basis has a unique dual Riesz basis.

Corollary 4.4. Suppose that *H* is a finitely or countably generated Hilbert *A*-module over a unital C*-algebra *A*. If *A* is commutative, then every Riesz basis of *H* has a unique dual Riesz basis.

Proof. Choose an arbitary Riesz basis $\{x_j\}_{j \in J}$ of *H***.** Suppose that $\{S^{-1} x_j + z_j\}_{j \in J}$ is a dual Riesz basi $\{x_j\}_{j \in J}$ s, where S is the frame operator of $\{x_j\}_{j \in J}$. Then by Theorem 4.3, for each $j \in J$ there exists $b_j \in A$ such that $z_j = b_j S^{-1} x_j$ and $\langle x, x_j \rangle b_j x_j = 0$ hold for all $x \in H$.

Since *A* is commutative, we have

$b_j \langle x, x_j \rangle x_j = 0$ for all $x \in H$ and $j \in J$.

Let $x = S^{-1} x_j$.

We have

$$0 = b_j \langle x, x_j \rangle x_j = b_j \langle s^{-1} x_j, x_j \rangle x_j =$$
$$b_j \langle x_j, s^{-1} x_j \rangle x_j = b_j x_j$$

Which yields that

$$z_j = b_j S^{-1} x_j = 0.$$

The following example show that the converse of corollary 4.4 is not true, namely, if every Riesz basis of a Hilber *A*-module of *H* has a unique dual Riesz basis, *A* is not necessarily commutative.

Example 4.5. Let $A = D_{2\times 2}(C)$ denote the C*-algebra of all 2×2 complex matrices. Let *H*=*A* and for any *A*,*B*∈*H* define

$$\langle A, B \rangle = AB^*.$$

Then *H* is a Hilbert *A*-module.
It is obvious *A* is commutative.
Let $E_{i,j}$ be the 2×2 matrix with 1 in the *(i,j)* th entry and 0 elsewhere, where $1 \le i,j \le 2$. Then $\{E_{1,1}, E_{2,2}\}$ is a Riesz basis of *H*, and so it has a unique dual Riesz basis which is itself.
But the dual frame of $\{E_{1,1}, E_{2,2}\}$ is not unique. For example , one can verify that $\{E_{1,1} + \alpha E_{2,2}, \beta E_{1,1} + E_{2,2}\}$ is also a dual frame of $\{E_{1,1}, E_{2,2}\}$ For any $\alpha, \beta \in C$.

Example 5.5. Let

$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \forall a \in \mathbb{C} \right\}$$

and

$$A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} : \forall a, b, c, d, e \in \mathbb{C} \right\}$$

For any *A*,*B*∈*H* we define

$$\langle A, B \rangle = AB^*$$

then *H* is a *A*-module.

Note that A is not commutative. Let

$$E_\alpha = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\{E_\alpha\}$ is a Riesz basis of H and any Riesz basis of H has the form of $\{E_\alpha\}$ some nonzero $\alpha \in \mathbb{C}$.

It is easy to check that every dual Riesz basis of $\{E_\alpha\}$ for each nonzero $\alpha \in \mathbb{C}$ is unique.

-*Acknowledgments
The authors would like to thank references for giving useful suggestions for the improvement of this paper.

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7/11/2014