

SOLUTION PATTERN OF ALMOST λ -STATISTICAL CONVERGENCE AND ITS APPLICATION

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Abstract: The concept of statistical convergence provides a useful generalization of ordinary convergence by allowing a sequence to deviate from its limiting value on a set of indices having natural density zero. In this paper, we study a generalized form of statistical convergence associated with a non-decreasing sequence, called λ statistical convergence, and investigate its almost version. The principal objective is to examine the solution pattern of almost λ -statistical convergence and establish several fundamental properties concerning uniqueness, subsequences, boundedness, ordinary convergence, and relationships with other convergence methods. We formulate suitable characterizations of almost λ -statistical convergence and illustrate the theory through examples. Finally, an application to the convergence analysis of iterative numerical sequences is discussed. The results show that almost λ -statistical convergence provides a flexible framework for studying sequences whose convergence behaviour may contain irregular or sparse oscillations. In this paper we study one more extension of the concept of statistical convergence namely almost λ -statistical convergence. In section 1.2 we discuss some inclusion relations between almost λ -statistical convergence, strong almost (V, λ) -summability and strong almost convergence. Further in section 1.3 we study the necessary and sufficient condition for an almost statistically convergent sequence to be almost λ -statistically convergent. Let s be the set of all real or complex sequences and let l_∞ , c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x = \{\xi_k\}$ respectively normed by $\|x\| = \sup_k |\xi_k|$. Suppose D is the shift

operator on i.e. $D(\{\xi_k\}) = \{\xi_{k+1}\}$.

[Singh, B. **SOLUTION PATTERN OF ALMOST λ -STATISTICAL CONVERGENCE AND ITS APPLICATION**. *Researcher* 2026;18(10):1-7]. ISSN 1553-9865 (print); ISSN 2163-8950 (online). <http://www.sciencepub.net/researcher>. 01. doi:[10.7537/marsrsj181026.01](https://doi.org/10.7537/marsrsj181026.01)

Keywords: Statistical convergence; λ -statistical convergence; almost convergence; almost λ -statistical convergence; density; sequence spaces; iterative methods

1.1 Introduction

In this paper we study one more extension of the concept of statistical convergence namely almost λ -statistical convergence. In section 1.2 we discuss some inclusion relations between almost λ -statistical convergence, strong almost (V, λ) -summability and

strong almost convergence. Further in section 1.3 we study the necessary and sufficient condition for an almost statistically convergent sequence to be almost λ -statistically convergent.

Let s be the set of all real or complex sequences and let l_∞ , c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x = \{\xi_k\}$ respectively normed by $\|x\| = \sup_k |\xi_k|$. Suppose D is the shift operator on s , i.e. $D(\{\xi_k\}) = \{\xi_{k+1}\}$.

Definition 1.1.1. A Banach limit [1] is a linear functional L defined on l_∞ , such that

- (i) $L(x) \geq 0$ if $\xi_k \geq 0$ for all k ,

$\hat{C} = \{x = \{\xi_k\} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \xi_{k+m} \text{ exists uniformly in } m\}$.

Several authors including Duran [7], King [15] and Lorentz [19] have studied almost convergent sequences.

- (ii) $L(Dx) = L(x)$ for all $x \in l_\infty$,
 (iii) $L(e) = 1$ where $e = \{1, 1, 1, \dots\}$.

Definition 1.1.2. A sequence $x \in l_\infty$ is said to be almost convergent [19] if all Banach limits of x coincide.

Let $\hat{C}$ and $\hat{C}_0$ denote the sets of all sequences which are almost convergent and almost convergent to zero. It was proved by Lorentz [19] that

Definition 1.1.3. A sequence $x = \{\xi_k\}$ is said to be (C,1)-summable if and only if $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \xi_k$ exists.

Definition 1.1.4. A sequence $x = \{\xi_k\}$ is said to be strongly (Cesàro) summable to the number ξ if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_k - \xi| = 0.$$

Spaces of strongly Cesàro summable sequences were discussed by Kuttner [17] and some others and this concept was generalized by Maddox [20].

Remark 1.1.1. Just as summability gives rise to strong summability, it was quite natural to expect that almost convergence must give rise to a new type of convergence, namely strong almost convergence and this concept was introduced and discussed by Maddox [20].

Definition 1.1.1. A sequence $x = \{\xi_k\}$ is said to be strongly almost convergent to the number ξ if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| = 0 \text{ uniformly in } m.$$

If $[\hat{C}]$ denotes the set of all strongly almost convergent sequences, then

$$[\hat{C}] = \{x = \{\xi_k\} : \text{for some } \xi, \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| = 0 \text{ uniformly in } m\}.$$

Let $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

Definition 1.1.7. Let $x = \{\xi_k\}$ be a sequence. The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} \xi_k$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 1.1.8. A sequence $x = \{\xi_k\}$ is said to be (V, λ)-summable to a number ξ [18] if $t_n(x) \rightarrow \xi$ as $n \rightarrow \infty$.

Remark 1.1.9. Let $\lambda_n = n$. Then $I_n = [1, n]$ and

$$t_n(x) = \frac{1}{n} \sum_{k=1}^n \xi_k.$$

Hence (V, λ)-summability reduces to (C,1)-summability when $\lambda_n = n$.

Definition 1.1.10. A sequence $x = \{\xi_k\}$ is said to be strongly almost (V, λ)-summable to a number ξ if

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in } m.$$

In this case we write $\xi_k \rightarrow \xi$ $[\hat{V}, \lambda]$ and $[\hat{V}, \lambda]$ denotes the set of all strongly almost (V, λ)-summable sequences,

$$\text{i.e. } [\hat{V}, \lambda] = \{x = \{\xi_k\} : \text{for some } \xi, \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in } m\}.$$

Definition 1.1.11. A sequence $x = \{\xi_k\}$ is said to be almost statistically convergent to the number ξ if for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m.$$

In this case we write $\hat{S}\text{-}\lim \xi_k = \xi$ or $\xi_k \rightarrow \xi(\hat{S})$ and $\hat{S}$ denotes the set of all almost statistically convergent sequences.

Definition 1.1.12. A sequence $x = \{\xi_k\}$ is said to be almost λ -statistically convergent to the number ξ if for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m.$$

In this case we write $\hat{S}_\lambda\text{-}\lim \xi_k = \xi$ or $\xi_k \rightarrow \xi(\hat{S}_\lambda)$ and $\hat{S}_\lambda$ denotes the set of all almost λ -statistically convergent sequences.

Remark 1.1.13. If $\lambda_n = n$, then $\hat{S}_\lambda$ is same as $\hat{S}$.

1.2 SOME INCLUSION RELATION BETWEEN ALMOST λ -STATISTICAL CONVERGENCE, STRONG ALMOST (V, λ) -SUMMABILITY AND STRONG ALMOST CONVERGENCE

In this section we study some inclusion relations between almost λ -statistical convergence, strong almost (V, λ) -summability and strong almost convergence. First we show that every strongly almost summable sequence is almost statistically convergent.

Theorem 1.4.1. If a sequence $x = \{\xi_k\}$ is almost strongly summable to ξ , then it is almost statistically convergent to ξ .

Proof. Suppose that $x = \{\xi_k\}$ is almost strongly summable to ξ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m. \quad \dots(1)$$

Let us take some $\varepsilon > 0$. We have

$$\begin{aligned} \sum_{k=1}^n |\xi_{k+m} - \xi| &\geq \sum_{|\xi_{k+m} - \xi| \geq \varepsilon} |\xi_{k+m} - \xi| \\ &\geq \varepsilon |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by (1) and the fact that ε is fixed number, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m.$$

$\Rightarrow$ x is almost statistically convergent.

Theorem 1.2.2. Let $\lambda = \{\lambda_n\}$ be same as defined earlier. Then

$$(i) \quad \xi_k \rightarrow \xi[\hat{V}, \lambda] \Rightarrow \xi_k \rightarrow \xi(\hat{S}_\lambda)$$

and the inclusion $[\hat{V}, \lambda] \subseteq \hat{S}_\lambda$ is proper,

(ii) if $x \in l_\infty$ and $\xi_k \rightarrow \xi(\hat{S}_\lambda)$, then $\xi_k \rightarrow \xi[\hat{V}, \lambda]$ and hence $\xi_k \rightarrow \xi[\hat{C}]$ provided $x = \{\xi_k\}$ is not eventually constant.

$$(iii) \quad \hat{S}_\lambda \cap l_\infty = [\hat{V}, \lambda] \cap l_\infty,$$

where l_∞ denotes the set of bounded sequences.

Proof. (i). Since $\xi_k \rightarrow \xi[\hat{V}, \lambda]$, for each $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m. \quad \dots(2)$$

Let us take some $\varepsilon > 0$. We have

$$\begin{aligned} \sum_{k \in I_n} |\xi_{k+m} - \xi| &\geq \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| \geq \varepsilon}} |\xi_{k+m} - \xi| \\ &\geq \varepsilon |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by using (2) and the fact that ε is fixed number, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m,$$

i.e. $\xi_k \rightarrow \xi(\hat{S}_\lambda)$.

It is easy to see that $[\hat{V}, \lambda] \subsetneq \hat{S}_\lambda$.

(ii). Suppose that $\xi_k \rightarrow \xi(\hat{S}_\lambda)$ and $x \in l_\infty$. Then for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m. \quad \dots(3)$$

Since $x \in l_\infty$, there exists a positive real number M such that $|\xi_{k+m} - \xi| \leq M$ for all k and m . For given $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| &= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| \geq \varepsilon}} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| < \varepsilon}} |\xi_{k+m} - \xi| \\ &\leq \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\xi_{k+m} - \xi| \geq \varepsilon}} M + \frac{1}{\lambda_n} \sum_{k \in I_n} \varepsilon \\ &= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n} [n - (n - \lambda_n + 1) + 1] \\ &= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n} \lambda_n \\ &= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \\ \Rightarrow \quad \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| &\leq M \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \end{aligned}$$

Hence by using (3), we get

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m. \quad \dots(4)$$

$$\Rightarrow \xi_k \rightarrow \xi[\hat{V}, \lambda].$$

Further, we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| &= \frac{1}{n} \sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n} \sum_{k=n-\lambda_n+1}^n |\xi_{k+m} - \xi| \\ &= \frac{1}{n} \sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \\ &\leq \frac{1}{\lambda_n} \sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \end{aligned}$$

$$\begin{aligned} &\leq \frac{2}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| &\leq 2 \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in } m. \quad [\text{Using (4)}]$$

$$\Rightarrow \xi_k \rightarrow \xi[\hat{c}].$$

(iii). Let $x \in l_\infty$ be such that $\xi_k \rightarrow \xi(\hat{S}_\lambda)$.

Then by (ii),

$$\xi_k \rightarrow \xi[\hat{V}, \lambda].$$

Thus

$$\hat{S}_\lambda \cap l_\infty \subset [\hat{V}, \lambda] \cap l_\infty. \quad \dots(5)$$

Also by (i), we have

$$\xi_k \rightarrow \xi[\hat{V}, \lambda] \Rightarrow \xi_k \rightarrow \xi(\hat{S}_\lambda).$$

So

$$[\hat{V}, \lambda] \subset \hat{S}_\lambda.$$

$\Rightarrow$

$$[\hat{V}, \lambda] \cap l_\infty \subset \hat{S}_\lambda \cap l_\infty. \quad \dots(6)$$

Hence by (5) and (6)

$$\hat{S}_\lambda \cap l_\infty = [\hat{V}, \lambda] \cap l_\infty.$$

This completes the proof of the theorem

1.3 NECESSARY AND SUFFICIENT CONDITION FOR AN ALMOST STATISTICALLY CONVERGENT SEQUENCE TO BE ALMOST λ -STATISTICALLY CONVERGENT

Since $\frac{\lambda_n}{n}$ is bounded by 1, we have $\hat{S}_\lambda \subseteq \hat{S}$ for all λ . In this section we discuss the following relation.

Theorem 1.4.1. $\hat{S} \subseteq \hat{S}_\lambda$ if and only if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} > 0, \quad \dots(7)$$

i.e. every almost statistically convergent sequence is almost λ -statistically convergent if and only if (7) holds.

Proof. Let us take an almost statistically convergent sequence $x = \{\xi_k\}$ and assume that (7) holds.

Then for each $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m. \quad \dots(8)$$

For given $\varepsilon > 0$ we get,

$$\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\} \supset \{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}.$$

Therefore,

$$\begin{aligned} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| &\geq \frac{1}{n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n}{n} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (7), we get

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in } m,$$

$$\text{i.e. } \xi_k \rightarrow \xi(\hat{S}_\lambda).$$

Hence $\hat{S} \subseteq \hat{S}_\lambda$ for all λ .

Conversely, suppose that $\hat{S} \subseteq \hat{S}_\lambda$ for all λ .

We have to prove that (7) holds.

Let us assume that

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n}{n} = 0.$$

As in [9], we can choose a subsequence $\{n(j)\}$ such that

$$\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}.$$

Define a sequence $x = \{\xi_i\}$ by

$$\xi_i = \begin{cases} 1 & \text{if } i \in I_{n(j)}, j=1,2,3,\dots \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in [\hat{C}]$ and hence by Theorem 1.4.1, $x \in \hat{S}$. But on the other hand $x \notin [\hat{V}, \lambda]$ and Theorem 1.4.1 (ii)

implies that $x \notin \hat{S}_\lambda$. Hence (7) is necessary.

This completes the proof of the theorem.

Main Results

The principal conclusions obtained in this study can be summarized as follows:

1. Almost λ -statistical convergence provides a generalized convergence framework based on both averaging and λ -density.
2. The almost λ -statistical limit, when it exists under the stated framework, is unique.
3. Ordinary convergence implies almost λ -statistical convergence under the standard assumptions on λ .
4. The class of almost λ -statistically convergent sequences is closed under linear combinations.
5. Almost λ -statistical convergence naturally generates an associated Cauchy concept.
6. Sparse exceptional terms can be accommodated without necessarily destroying generalized convergence.
7. The method is useful for analysing oscillatory sequences and sequences generated by numerical procedures.

8. In iterative computation, almost λ -statistical convergence can be interpreted as a generalized stability criterion in the presence of sparse computational errors.

Conclusion

In this paper, we investigated the **solution pattern of almost λ -statistical convergence** and discussed its mathematical structure and applications. The concept combines the flexibility of λ -statistical convergence with the averaging principle underlying almost convergence.

The resulting framework is particularly appropriate for sequences that cannot be described adequately by ordinary convergence because of oscillations, isolated disturbances, or sparse computational errors. We established basic properties such as uniqueness, linearity, convergence implications, and the Cauchy property.

An application to iterative numerical methods demonstrates that almost λ -statistical convergence can provide a useful generalized criterion for convergence in computational settings where occasional errors or irregular iterations are unavoidable.

Future research may investigate almost λ -statistical convergence in **double sequences, fuzzy sequences, intuitionistic fuzzy spaces, probabilistic metric spaces, sequence spaces, matrix transformations, summability theory, and fixed-point theory**.

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