An Approach To The Erlang Loss System

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Abstract: A stochastic model is said to be *insensitive* if its stationary distribution depends on one or more of its constituent lifetime distributions only through the mean. In this paper we shall discuss insensitivity by presenting a detailed analysis of the canonical insensitive queueing model, by the Erlang loss system.

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Introduction

We shall start our discussion of insensitivity by thinking about the M/M/C/C (or Erlang Loss) queue. This is a queueing system which has Poisson arrivals, exponential service times, C servers and no room for queueing customers that arrive when the system is full. The queue can be modelled by a continuous-time Markov chain with state space {0,1,2, . . . ,C}. If we denote the arrival rate by λ and the mean service time by $\frac{1}{\mu}$, then the stationary probability $\pi(n)$ that there are n customers present satisfies the equations

l p (0) = m p (1),

(l +nm) p (n) = l p (n−1)+(n+1)m p (n+1), 0 < n <C,

Cm p (C) = l p (C−1). ***(1.1)***

The solution of equations (1.1) that sums to unity is

$$\pi(n) = \frac{\frac{\rho^n}{n!}}{\sum_{k=0}^{C}\frac{\rho^k}{k!}} \qquad (1.2)$$

Where $\rho = \frac{\lambda}{\mu}$. The stationary probability

$$\pi(C) = \frac{\frac{\rho^C}{C!}}{\sum_{k=0}^{C}\frac{\rho^k}{k!}} \qquad (1.3)$$

That the system is full gives the probability that arriving customers cannot be accommodated in the queue.

Expressed as a function of ρ and C, the expression on the right hand side of equation (1.3) is known as Erlang's Loss Formula, which we shall denote by $E(\rho, C)$. Throughout most of the twentieth century, this formula was used extensively by the telecommunications networking community for dimensioning links.

However, let us think a little more about the use of a Markovian model for the modelling of telephone links. The average duration of a traditional phone conversation was three minutes. An easy calculation shows that if call durations are exponentially distributed with mean three minutes, then the probability that a call exceeds 60 minutes is about 2×10^{-9}. So, if call durations really were exponentially distributed, very few of us would ever have made a phone call that lasted longer than one hour. Since most of us have made such calls, we are led to the conclusion that the 'service times' corresponding to real telephone conversations are not exponentially distributed and that a Markovian model for the system is based upon assumptions that are not satisfied.

So why has the Erlang Loss Formula been so successful? The reason is that the ***M/G/C/C*** queue is *insensitive* to the service time distribution: the stationary probability that there are n customers present is given by (1.2) irrespective of the shape of the service time distribution, provided that the mean is $\frac{1}{\mu}$.

Erlang himself [9] noticed that the stationary probability that there are n customers present in an ***M/G/C/C*** queue when the service times are deterministic with duration $\frac{1}{\mu}$ is the same as it is when service times are exponentially distributed with mean $\frac{1}{\mu}$. Subsequently, with different levels of rigour, Kosten [4], Fortet [3] and Sevastyanov [7] showed that the service time distribution can be arbitrary without affecting the form of the stationary probabilities, assuming that the mean is kept constant. In fact more is possible: service times can be inter-event times in an arbitrary stationary point process with rate μ and the stationary distribution is still given by (1.2), see K¨onig and Matthes [10]. Other authors who considered the *n* server loss system with generally distributed service times from the point of view of insensitivity include Takacs [8] who investigated the stationary distribution at arrival epochs and Fakinos [2] who looked at a group arrival, group departure system.

Baskett, Chandy, Muntz and Palacios [5] considered a network of queues where each node could be one of four different types. These were:

1. A single server, first-come-first-served queue with exponential service times,
2. A single server, processor-sharing queue with service times chosen according to a general distribution with a rational Laplace Transform,
3. An infinite-server queue with service times chosen according to a general distribution with a rational Laplace Transform, and
4. A single-server, preemptive-resume last-come-first-served queue with service times again chosen according to a general distribution with a rational **Laplace Transform.**

They showed that the queueing network possesses a steady state distribution that is a product form over the nodes and, moreover, depends on the lifetime distribution at types (2), (3) and (4) nodes only through the mean. Weak continuity arguments later showed that the restriction to distributions with rational Laplace transform was unnecessary, although many later papers continued to emphasize this restriction.

Kelly [11, 12] introduced the concept of the *symmetric* queue. This can be thought of as a generalization of the type (2), (3) and (4) nodes of [4]. A symmetric queue is a queue with multiple customer classes that operates in the following manner:

1. The service requirement of a customer is a random variable whose distribution may depend on the class of customer.
2. The total service effort is supplied at rate $f(n)$ where **n** is the number of customers in the queue.
3. A proportion $g(\ell,n)$ of this effort is directed to the customer in position ℓ. When this customer leaves the queue customers in positions $\ell+1, \ell+2, \ldots, n$ move to positions $\ell, \ell+1, \ldots, n-1$ respectively.
4. A customer arriving at the queue moves into position ℓ with probability $g(\ell,n+1)$. Customers previously in positions $\ell, \ell+1, \ldots, n$ move to positions $\ell+1, \ell+2, \ldots, n+1$ respectively.

Processor sharing queues, infinite server queues and last come first served queues are all examples of symmetric queues. By keeping track of the current 'phase' of service, Kelly showed that a stationary symmetric queue is insensitive to the service time distribution, provided that it can be represented as a mixture of Erlang distributions. The rigorous extension to arbitrarily distributed lifetimes was carried out by Barbour [1]. Furthermore, Kelly established that a network of symmetric queues has a stationary distribution that factorizes into a product form over the nodes, and itself is insensitive.

Conclusion

In this paper, we have presented an introduction to insensitivity as it occurs in stochastic models. Our approach has been to illustrate the main ideas using simple special cases.

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